Famous Classical Inequalities [1]

THE AM-GM INEQUALITY

Elementary Form. For any real numbers $x_1, x_2, ..., x_n \geq 0$,

$$\frac{x_1 + x_2 + ... + x_n}{n} \geq \sqrt[n]{x_1 x_2 ... x_n},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Weighted AM-GM Inequality. For any list of non-negative real numbers $w_1, w_2, ..., w_n$ satisfying $w_1 + w_2 + ... + w_n = w$, then for $x_1, x_2, ..., x_n \geq 0$,

$$\frac{w_1 x_1 + w_2 x_2 + ... + w_n x_n}{w} \geq \sqrt[w]{x_1^{w_1} x_2^{w_2} ... x_n^{w_n}},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

THE MEAN INEQUALITY CHAIN

The RMS-AM-GM-HM Chain. For any real numbers $x_1, x_2, ..., x_n \geq 0$, define

$$\text{RMS} = \sqrt{\frac{x_1^2 + x_2^2 + ... + x_n^2}{n}}, \quad \text{AM} = \frac{x_1 + x_2 + ... + x_n}{n},$$
$$\text{GM} = \sqrt[n]{x_1 x_2 ... x_n}, \quad \text{HM} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

Then, the following chain of inequalities holds true

$$\text{RMS} \geq \text{AM} \geq \text{GM} \geq \text{HM},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Remark. RMS-AM-GM-HM respectively stands for Root Mean Square, Arithmetic Mean, Geometric Mean, and Harmonic Mean.

THE POWER MEAN INEQUALITY

The Power Mean Inequality. The power mean inequality is the general case of the mean inequality chain. The power mean $M(p)$ is defined as:

$$M(p) = \begin{cases} \left(\frac{x_1^p + x_2^p + ... + x_n^p}{n}\right)^{1/p} & \text{if } p \neq 0 \\ \sqrt[n]{x_1 x_2 ... x_n} & \text{if } p = 0 \end{cases}$$

The Power Mean Inequality states that if $a > b$, then $M(a) > M(b)$, with equality holding if and only if $x_1 = x_2 = \cdots = x_n$.

Plugging in $p = 1, 0$ into this inequality reduces it to AM-GM, and $p = 2, 1, 0, -1$ gives the Mean Inequality Chain. As with AM-GM, a weighted version of the Power Mean Inequality also exists.

The Weighted Power Mean. Let $x_1, x_2, ..., x_n$ and $w_1, w_2, ..., w_n$ be positive real numbers with $w_1 + w_2 + ... + w_n = 1$. For any real number p, we define

$$M(p) = \begin{cases} \left(w_1 x_1^p + w_2 x_2^p + ... + w_n x_n^p\right)^{1/p} & \text{if } p \neq 0 \\ x_1^{w_1} x_2^{w_2} ... x_n^{w_n} & \text{if } p = 0 \end{cases}$$

If $a > b$, then $M(a) > M(b)$.

In particular, if $w_1 = w_2 = \cdots = w_n = \dfrac{1}{n}$, the above $M(p)$ is just the regular power mean inequality.

THE CAUCHY-SCHWARZ INEQUALITY

Elementary Form (Buniakovsky Form). For any list of real numbers $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$, the following inequality holds true

$$(a_1^2 + ... + a_n^2)(b_1^2 + ... + b_n^2) \geq (a_1 b_1 + ... + a_n b_n)^2,$$

with equality if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

In other words, there exists some $t \in \mathbb{R}$ such that $a_i = t b_i,\ 0 < i \leq n$, or if every number in one of the list is zero.

Engel Form (or Titu Lemma/Bergström's Inequality). For any list of real numbers $a_1, a_2, ..., a_n$ and positive reals $b_1, b_2, ..., b_n$, we have

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + ... + a_n)^2}{b_1 + b_2 + ... + b_n},$$

with equality if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

Weighted Cauchy-Schwarz Inequality. Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be any lists of real numbers, and let $w_1, w_2, ..., w_n$ be positive real numbers. Then we have the inequality

$$\left(\sum_{i=1}^n a_i^2 w_i\right)\left(\sum_{i=1}^n b_i^2 w_i\right) \geq \left(\sum_{i=1}^n a_i b_i w_i\right)^2,$$

with equality if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

Complex Form. Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be two list of complex numbers. Then,

$$\left(\sum_{i=1}^n |a_i^2|\right)\left(\sum_{i=1}^n |b_i^2|\right) \geq \left|\sum_{i=1}^n a_i b_i\right|^2,$$

This appears to be more powerful, but it follows from

$$\left(\sum_{i=1}^n |a_i^2|\right)\left(\sum_{i=1}^n |b_i^2|\right) \geq \left(\sum_{i=1}^n |a_i| \cdot |b_i|\right)^2 \geq \left|\sum_{i=1}^n a_i b_i\right|^2.$$

Again, equality occurs if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

Inner Product Form (General Form). The Cauchy-Schwarz Inequality can be generalized to the inner product space. Let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in an inner product space. Then,

$$|\langle \mathbf{u}, \mathbf{v} \rangle|^2 \leq \|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle \cdot \langle \mathbf{v}, \mathbf{v} \rangle,$$

where $\langle \ , \ \rangle$ is the inner product, and $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$ is the norm of $\mathbf{u}$.

In particular, if we let $\mathbf{u} = (u_1, u_2, ..., u_n)$, $\mathbf{v} = (v_1, v_2, ..., v_n)$ and define $\langle \ , \ \rangle$ to be the dot product on $\mathbb{R}^n$, then for any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}|^2 \leq \|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2 \quad \text{or,} \quad |\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|.$$

with equality if and only if there exists and scalar t such that $\mathbf{u} = t\mathbf{v}$, or if one of the vectors is the zero vector. If we write down all the components of $\mathbf{u}$, and $\mathbf{v}$, then we obtain the real form of the C-S inequality.

On the other hand, if we define $\langle \mathbf{u}, \mathbf{v} \rangle := u_1 \overline{v_1} + u_2 \overline{v_2} + ... + u_n \overline{v_n}$ on $\mathbb{C}^n$, then this is just the previously stated complex form of the C-S inequality.

Integral Form. If instead we let $\mathbf{u} = f(x)$, $\mathbf{v} = g(x)$ be vectors in the space of continuous functions $C[a, b]$, and define $\langle \ , \ \rangle$ to be

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle f(x), g(x) \rangle = \int_a^b f(x) g(x) \, dx,$$

then,

$$\left(\int_a^b f(x) g(x)\right)^2 \leq \left(\int_a^b f^2(x)\right)\left(\int_a^b g^2(x)\right),$$

with equality if and only if $f(x)$ and $g(x)$ are linearly independent, meaning that $f(x) = k g(x)$, for some $k \in \mathbb{R}$.

HOLDER'S INEQUALITY

Elementary Form. Given two sequences of non-negative real numbers $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$, and let $p, q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$(a_1^p + ... + a_n^p)^{\frac{1}{p}} \cdot (b_1^q + ... + b_n^q)^{\frac{1}{q}} \geq a_1 b_1 + ... + a_n b_n$$

or,

$$\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}} \geq \sum_{i=1}^n a_i b_i,$$

with equality if and only if $\dfrac{a_1^p}{b_1^q} = \dfrac{a_2^p}{b_2^q} = \cdots = \dfrac{a_n^p}{b_n^q}$.

Weighted Form. Given $a_1, a_2, ..., a_n$; $b_1, b_2, ..., b_n$, and $w_1, w_2, ..., w_n$ be three sequences of non-negative real numbers, and let $p, q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left(\sum_{i=1}^n a_i^p w_i\right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^p w_i\right)^{\frac{1}{q}} \geq \sum_{i=1}^n a_i b_i w_i,$$

with equality if and only if $\dfrac{a_1^p}{b_1^q} = \dfrac{a_2^p}{b_2^q} = \cdots = \dfrac{a_n^p}{b_n^q}$.

[1] This note was written by ddj1572 [Discord].

Concise Form. Let a_{ij} be non-negative real numbers (with $i = \overline{1;m}$ and $j = \overline{1;n}$) and let $\{\lambda\}_{i=1}^n$ be a sequence of non-negative reals such that $\sum_{1\le i\le n} \lambda_i = \lambda_1 + \lambda_2 + ... + \lambda_n = 1$. Then

$$(a_{11} + ... + a_{1n})^{\lambda_1}(a_{21} + ... + a_{2n})^{\lambda_2}...(a_{m1} + ... + a_{mn})^{\lambda_n} \ge \left(a_{11}^{\lambda_1} a_{21}^{\lambda_2} ... a_{m1}^{\lambda_n}\right) ... \left(a_{1n}^{\lambda_1} a_{2n}^{\lambda_2} ... a_{mn}^{\lambda_n}\right)$$

or,

$$\left(\sum_{i=1}^n a_{1i}\right)^{\lambda_1}\left(\sum_{i=1}^n a_{2i}\right)^{\lambda_2}\cdots\left(\sum_{i=1}^n a_{mi}\right)^{\lambda_n} \ge \sum_{i=1}^n \left(a_{1i}^{\lambda_1} a_{2i}^{\lambda_2} ... a_{mi}^{\lambda_n}\right)$$

or,

$$\prod_{i=1}^m \left(\sum_{j=1}^n a_{ij}\right)^{\lambda_i} \ge \sum_{j=1}^n \left(\prod_{i=1}^m a_{ij}^{\lambda_i}\right).$$

Note that with only two sequences $\{a_1\}$ and $\{a_2\}$, and $\lambda_1 = \lambda_2 = 1/2$, this is the elementary form of the Cauchy-Schwarz Inequality.

Integral Form. Given real numbers $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and let f, g be two continuous functions on $[a, b]$. Then the following inequality holds:

$$\int_a^b |f(x)\cdot g(x)|dx \le \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}},$$

with equality if and only if there exists real number α and β such that $\alpha^2 + \beta^2 > 0$ and $\alpha|f(x)|^p = \beta|g(x)|^q$, $\forall x \in [a, b]$. That is, $|f|^p$ and $|g|^q$ are linearly independent.

General Form. Let (S, Σ, μ) be a measure space, and let $p, q \in [1, \infty) \cup \{\infty\}$ be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for all measurable real, or complex values functions f and g on S,

$$\|f\ g\|_1 \ge \|f\|_p\ \|g\|_q.$$

If $p, q \in (1, \infty)$, and $f, g \in L^p(\mu)$, then Holder's inequality becomes an equality if and only if $|f|^p$ and $|g|^q$ are linearly independent in $L^1(\mu)$. That is, there exists real number α and β such that $\alpha^2 + \beta^2 > 0$ and $\alpha|f|^p = \beta|g|^q$.

MINKOWSKI'S INEQUALITY

The First Minkowski's Inequality. Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be positive real numbers and let $p > 0$. Then,

$$\left(\sum_{i=1}^n (a_i + b_i)^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n b_i^p\right)^{\frac{1}{p}},$$

with equality if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

The Second Minkowski's Inequality. Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be positive real numbers and let $p > 0$. Then,

$$\left[\left(\sum_{i=1}^n a_i^p\right)^p + \left(\sum_{i=1}^n b_i^p\right)^p\right]^{\frac{1}{p}} \le \sum_{i=1}^n (a_i^p + b_i^p)^{\frac{1}{p}},$$

with equality if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

The Third Minkowski's Inequality. Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be positive real numbers. Then,

$$\sqrt[n]{a_1 a_2 ... a_n} + \sqrt[n]{b_1 b_2 ... b_n} \le \sqrt[n]{(a_1 + b_1)(a_2 + b_2)...(a_n + b_n)},$$

with equality if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

Weighted Minkowski Inequality. Given $a_1, a_2, ..., a_n$; $b_1, b_2, ..., b_n$, and $w_1, w_2, ..., w_n$ be three sequences of non-negative real numbers, and let $p > 0$. Then,

$$\left(\sum_{i=1}^n (a_i + b_i)^p w_i\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^n a_i^p w_i\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n b_i^p w_i\right)^{\frac{1}{p}},$$

with equality if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

General Form. For non-negative real numbers a_{ij}, ($i = \overline{1;m}$ and $j = \overline{1;n}$), and for $r > s$,

$$\left(\sum_{i=1}^m \left(\sum_{i=1}^n a_{ij}^r\right)^{s/r}\right)^{\frac{1}{s}} \ge \left[\sum_{j=1}^n \left(\sum_{i=1}^m a_{ij}^s\right)^{r/s}\right]^{\frac{1}{r}}, \ \forall p > s$$

Integral Form. Given $p > 1$ and let f, g be continuous functions on $[a, b]$. Then,

$$\left(\int_a^b |f(x) + g(x)|^p dx\right)^{1/p} \le \left(\int_a^b |f(x)|^p dx\right)^{1/p} + \left(\int_a^b |g(x)|^p dx\right)^{1/p}$$

General Form. Let S be a measure space, let $p \in [1, \infty)$, and let $f \in L^p(S)$ and $g \in L^p(S)$. Then, $f + g \in L^p(S)$, and we have

$$\|f + g\|_p \le \|f\|_p + \|g\|_p,$$

with inequality for $p \in (1, \infty)$ if and only if f and g are linearly dependent. That is, $f = kg$, for some $k \ge 0$, or $g = 0$.

REARRANGEMENT INEQUALITY

The Rearrangement Inequality. Given two sequences of real numbers $\{a_i\}_{i=1}^n$ satisfying $a_1 \ge a_2 \ge ... \ge a_n$ and $\{b_i\}_{i=1}^n$.

1. If $b_1 \ge b_2 \ge ... \ge b_n$, then
$$a_1b_1 + a_2b_2 + ... + a_nb_n \ge a_1b_{\sigma(1)} + a_2b_{\sigma(2)} + ... + a_nb_{\sigma(n)} \ge a_1b_n + a_2b_{n-1} + ... + a_nb_1.$$
2. If $b_1 \le b_2 \le ... \le b_n$, then
$$a_1b_1 + a_2b_2 + ... + a_nb_n \le a_1b_{\sigma(1)} + a_2b_{\sigma(2)} + ... + a_nb_{\sigma(n)} \le a_1b_n + a_2b_{n-1} + ... + a_nb_1,$$

Where $\sigma(1), \sigma(2), ..., \sigma(n)$ is any permutation of $1, 2, ..., n$.

Lemma. Given a sequence of real numbers $\{x_i\}_{i=1}^n$ and $x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}$ is any permutation of $(x_1, x_2, ..., x_n)$. Then,

1. $x_1^2 + x_2^2 + ... + x_n^2 \ge x_1x_{\sigma(1)} + x_2x_{\sigma(2)} + ... + x_nx_{\sigma(n)}$,
2. $\frac{x_{\sigma(1)}}{x_1} + \frac{x_{\sigma(2)}}{x_2} + ... + \frac{x_{\sigma(n)}}{x_n} \ge n$.

The Reverse Rearrangement Inequality. Given two sequences of real numbers $\{a_i\}_{i=1}^n$ satisfying $a_1 \ge a_2 \ge ... \ge a_n$ and $\{b_i\}_{i=1}^n$.

1. If $b_1 \ge b_2 \ge ... \ge b_n$, then
$$\prod_{k=1}^n (a_k + b_k) \le \prod_{k=1}^n (a_k + b_{\sigma(k)}) \le \prod_{k=1}^n (a_k + b_{n-k+1}).$$
2. If $b_1 \le b_2 \le ... \le b_n$, then
$$\prod_{k=1}^n (a_k + b_k) \ge \prod_{k=1}^n (a_k + b_{\sigma(k)}) \ge \prod_{k=1}^n (a_k + b_{n-k+1}),$$

where $\sigma(1), \sigma(2), ..., \sigma(n)$ is any permutation of $1, 2, ..., n$.

CHEBYSHEV'S INEQUALITY

Elementary Form. Given two sequences of real numbers $\{a_i\}_{i=1}^n = \{a_1, a_2, ..., a_n\}$ satisfying $a_1 \ge a_2 \ge ... \ge a_n$ and $\{b_i\}_{i=1}^n = \{b_1, b_2, ..., b_n\}$.

1. If $b_1 \ge b_2 \ge ... \ge b_n$, then
$$n\sum_{i=1}^n a_ib_i \ge \left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right) \ge n\sum_{i=1}^n a_ib_{n+1-i},$$
2. If $b_1 \le b_2 \le ... \le b_n$, then
$$n\sum_{i=1}^n a_ib_i \le \left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right) \le n\sum_{i=1}^n a_ib_{n+1-i},$$

with equality if and only if $a_1 = a_2 = ... = a_n$ or $b_1 = b_2 = ... = b_n$.

Weighted Form. Given two sequences of real numbers $\{a_i\}_{i=1}^n = \{a_1, a_2, ..., a_n\}$ satisfying $a_1 \ge a_2 \ge ... \ge a_n$ and $\{b_i\}_{i=1}^n = \{b_1, b_2, ..., b_n\}$. Let $w_1, w_2, ..., w_n$ be non-negative real numbers such that $w_1 + w_2 + ... + w_n = 1$.

1. If $b_1 \ge b_2 \ge ... \ge b_n$, then
$$n\sum_{i=1}^n a_ib_iw_i \ge \left(\sum_{i=1}^n a_iw_i\right)\left(\sum_{i=1}^n b_iw_i\right) \ge n\sum_{i=1}^n a_ib_{n+1-i}w_i,$$
2. If $b_1 \le b_2 \le ... \le b_n$, then
$$n\sum_{i=1}^n a_ib_iw_i \le \left(\sum_{i=1}^n a_iw_i\right)\left(\sum_{i=1}^n b_iw_i\right) \le n\sum_{i=1}^n a_ib_{n+1-i}w_i,$$

with equality if and only if $a_1 = a_2 = ... = a_n$ or $b_1 = b_2 = ... = b_n$.

Integral Form. If $f(x)$ and $g(x)$ are two increasing functions, or two decreasing functions, then

$$\frac{1}{b-a}\int_a^b f(x)\cdot g(x)dx \ge \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)$$

If $f(x)$ and $g(x)$ are oppositely monotonic, meaning that one function is increasing, one function is decreasing, then

$$\frac{1}{b-a}\int_a^b f(x)\cdot g(x)dx \le \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)$$

Since these are the recognizable averages of functions, we can rewrite these inequalities as:

$$\begin{cases}(f\cdot g)_{avg} \ge f_{avg}\cdot g_{avg} & \text{if } f(x), g(x) \text{ are identically monotonic}\\ (f\cdot g)_{avg} \le f_{avg}\cdot g_{avg} & \text{if } f(x), g(x) \text{ are oppositely monotonic}\end{cases}$$

BERNOULLI'S INEQUALITY

Classical Form. Let $x \geq -1$ be real numbers, then

$$\begin{cases} (1+x)^r \geq 1 + rx & \text{if } r \geq 1 \text{ or } r \leq 0 \\ (1+x)^r \leq 1 + rx & \text{if } 0 \leq r \leq 1 \end{cases}$$

Lemma. Let t and α be real numbers, then

$$\begin{cases} t^\alpha \geq \alpha t + (1-\alpha) & , \forall t > 0,\ \alpha \in (-\infty, 0] \cup [1, \infty) \\ t^\alpha \leq \alpha t + (1-\alpha) & , \forall t > 0,\ \alpha \in [0, 1] \end{cases}$$

Primitive Form. Let $x_1, x_2, ..., x_n$ be real numbers of the same signs and greater than -1, then

$$(1+x_1)(1+x_2)...(1+x_n) \geq 1 + (x_1 + x_2 + ... + x_n).$$

YOUNG'S INEQUALITY

Elementary Form. Let a, b be non-negative real numbers, and p, q be positive reals that satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab \quad (1), \quad \text{or} \quad \frac{a}{p} + \frac{b}{q} \geq a^{1/p} \cdot b^{1/q} \quad (2),$$

with equality if and only if $a^p = b^q$ for (1) and $a = b$ for (2).

For $p < 1, p \neq 0$, we have

$$\frac{a^p}{p} + \frac{b^q}{q} \leq ab, \quad \text{or} \quad \frac{a}{p} + \frac{b}{q} \leq a^{1/p} \cdot b^{1/q},$$

with equality if and only if $a^p = b^q$ or $a = b$.

General Form. Given a sequence of non-negative real numbers $a_1, a_2, ..., a_n$, and $p_1, p_2, ..., p_n > 0$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_n} = 1$. Then,

$$\frac{a_1}{p_1} + \frac{a_2}{p_2} + ... + \frac{a_n}{p_n} \geq a_1^{1/p_1} \cdot a_2^{1/p_2} ... a_n^{1/p_n},$$

with equality if and only if $a_1 = a_2 = ... = a_n$.

By letting $p_1 = p_2 = ... = p_n = n$, we obtain the familiar AM-GM inequality.

Integral Form (For Increasing Functions). Let $f : X \to Y$ be a continuous, increasing function defined for non-negative real number x, with $f(0) = 0$. Suppose a, b are positive reals such that $a \in X = D(f)$ and $b \in Y$. Then,

$$\int_a^b f(x)dx + \int_a^b f^{-1}(x)dx \geq ab,$$

with equality if and only if $f(a) = b$.

SCHUR'S INEQUALITY

Elementary Form. Let a, b, c be non-negative real numbers and let r be some arbitrary positive integer. Then

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0,$$

with equality if and only if $a = b = c$; or $a = 0, b = c$, vice versa.

General Form. Consider $a, b, c, x, y, z \in \mathbb{R}$, where $a \geq b \geq c$, and either $x \geq y \geq z$ or $x \leq y \leq z$ is true. Let $k \in \mathbb{Z}^+$, and let $f : \mathbb{R} \to \mathbb{R}_0^+$ be either a convex or monotonic function on $\mathbb{R}$. Then

$$f(x)(a-b)^k(a-c)^k + f(y)(b-c)^k(b-a)^k + f(z)(c-a)^k(c-b)^k \geq 0,$$

with equality if and only if $a = b = c$; or $a = 0, b = c$, vice versa.

MUIRHEAD'S INEQUALITY

Symmetric Summation. Consider a function of n variables $f(x_1, x_2, ..., x_n)$. The *symmetric summation* is defined to be

$$T(a) = T(x, a) = \sum_{sym} f(x_1, x_2, ..., x_n) = \sum_{\sigma \in S_{(n)}} f(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)})$$

where $\sigma(1), \sigma(2), ..., \sigma(n)$ is any permutation of $1, 2, ..., n$ and $S_{(n)}$ is the set of all permutation of $\{1, 2, ..., n\}$.

Examples.

$$\sum_{sym} \frac{x^3}{y^2 - z^5} = \frac{x^3}{y^2 - z^5} + \frac{x^3}{z^2 - y^5} + \frac{y^3}{x^2 - z^5} + \frac{y^3}{z^2 - x^5} + \frac{z^3}{x^2 - y^5} + \frac{x^3}{y^2 - x^5}$$

$$\sum_{sym} x^{y-z} = x^{y-z} + x^{z-y} + y^{x-z} + y^{z-x} + z^{x-y} + z^{y-x}$$

For a non-negative sequence $\{a_i\}_{i=1}^n$ and a set of positive real numbers $\{x_1, x_2, ..., x_n\}$, the *symmetric sum* of $x_1^{a_1} x_2^{a_2} ... x_n^{a_n}$ is the expression

$$T(a) = T(x, a) = \sum_{sym} x_1^{a_1} x_2^{a_2} ... x_n^{a_n} = \sum_{\sigma \in S_{(n)}} x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} ... x_{\sigma(n)}^{a_n},$$

where $\sigma(1), \sigma(2), ..., \sigma(n)$ is any permutation of $1, 2, ..., n$ and $S_{(n)}$ is the set of all permutation of $\{1, 2, ..., n\}$.

Examples.

$$\begin{aligned} \sum_{sym} x^3 y^2 z^0 &= x^3 y^2 z^0 + x^3 z^2 y^0 + y^3 x^2 z^0 + y^3 z^2 x^0 + z^3 x^2 y^0 + z^3 y^2 x^0 \\ &= x^3 y^2 + x^3 z^2 + y^3 x^2 + y^3 z^2 + z^3 x^2 + z^3 y^2 \end{aligned}$$

Majorization. We say a non-increasing sequence of number $\{a_i\}_{i=1}^n$ majorizes another non-increasing sequence $\{b_i\}_{i=1}^n$, and write $\{a_i\}_{i=1}^n \succ \{b_i\}_{i=1}^n$ if and only if for all $1 \leq k \leq n$, $\sum_{i=1}^k a_i \geq \sum_{i=1}^k b_i$, with equality when $k = n$. Symbolically,

$$\{a_1, a_2, ..., a_n\} \succ \{b_1, b_2, ..., b_n\}$$

if and only if

$$\begin{cases} a_1 \geq b_1 \\ a_1 + a_2 \geq b_1 + b_2 \\ \dots\dots\dots\dots\dots\dots \\ a_1 + a_2 + ... + a_{n-1} \geq b_1 + b_2 + ... + b_{n-1} \\ a_1 + a_2 + ... + a_{n-1} + a_n = b_1 + b_2 + ... + b_{n-1} + b_n \end{cases}$$

If $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ are not necessarily non-increasing, then we still write $\{a_i\}_{i=1}^n \succ \{b_i\}_{i=1}^n$ if this is true after the sequences have been sorted in non-increasing order.

Elementary Form. Consider two pairs of positive real numbers $\{a_1, a_2\}$ and $\{b_1, b_2\}$. Consider also, positive real numbers x, y. If $\{a_1, a_2\} \succ \{b_1, b_2\}$, then

$$\sum_{sym} x^{a_1} y^{a_2} \geq \sum_{sym} x^{b_1} y^{b_2}$$

with equality if and only if $a_1 = b_1, a_2 = b_2$; or $x = y$.

General Form. Consider two sequences of positive real numbers $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$. Consider also, a set of n positive real numbers $\{x_1, x_2, ..., x_n\}$. If $\{a_i\}_{i=1}^n \succ \{b_i\}_{i=1}^n$, then

$$\sum_{sym} x_1^{a_1} x_2^{a_2} ... x_n^{a_n} \geq \sum_{sym} x_1^{b_1} x_2^{b_2} ... x_n^{b_n},$$

with equality if and only if $a_i = b_i, i = \overline{1;n}$; or $x_1 = x_2 = ... = x_n$.

JENSEN'S INEQUALITY

Convexity and Concavity of a Function. Suppose that f is a one-variable function defined on $[a, b] \subseteq \mathbb{R}$. The function f is called a convex function on $[a, b]$ if and only if $\forall x, y \in [a, b]$, and $\forall t \in [0, 1]$, we have

$$tf(x) - (1-x)f(y) \geq f(tx + (1-t)y)$$

Similarly, a function f is called a concave function on $[a, b]$ if

$$tf(x) - (1-x)f(y) \leq f(tx + (1-t)y)$$

Elementary Form. Let $x_1, x_2, ..., x_n \in I \subseteq \mathbb{R}$, and let f be a function of one variable defined on I.

1. If f is a convex function on I, then
$$f\left(\frac{x_1 + x_2 + ... + x_n}{n}\right) \leq \frac{f(x_1) + f(x_2) + ... + f(x_n)}{n}.$$
2. If f is a concave function on I, then
$$f\left(\frac{x_1 + x_2 + ... + x_n}{n}\right) \geq \frac{f(x_1) + f(x_2) + ... + f(x_n)}{n},$$

with equality if and only if $x_1 = x_2 = ... = x_n$.

General Form. Let f be a function of one variable defined on $I \subseteq \mathbb{R}$. Also let $x_1, x_2, ..., x_n \in I \subseteq \mathbb{R}$ and $\alpha_1, \alpha_2, ..., \alpha_n \geq 0$ such that $\alpha_1 + \alpha_2 + ... + \alpha_n = 1$.

1. If f is a convex function on I, then
$$f(\alpha_1 x_1 + ... + \alpha_n x_n) \leq \alpha_1 f(x_1) + ... + \alpha_n f(x_n).$$
2. If f is a concave function on I, then
$$f(\alpha_1 x_1 + ... + \alpha_n x_n) \geq \alpha_1 f(x_1) + ... + \alpha_n f(x_n),$$

with equality if and only if $x_1 = x_2 = ... = x_n$.

KARAMATA'S INEQUALITY

Given two sequences (or lists) of numbers $\{a_i\}_{i=1}^n = \{a_1, a_2, ..., a_n\}$ and $\{b_i\}_{i=1}^n = \{b_1, b_2, ..., b_n\}$ such that $a_i, b_i \in I \subseteq \mathbb{R}$. If $\{a_i\} \succ \{b_i\}$ ($\{a_i\}$ majorizes $\{b_i\}$), and f is a convex function on $I \subseteq \mathbb{R}$, then

$$\sum_{i=1}^n f(a_i) \geq \sum_{i=1}^n f(b_i).$$

POPOVICIU'S INEQUALITY

Elementary Form. Let f be a function defined on an interval $I \subseteq \mathbb{R}$.

1. If f is convex, then for any three points x, y, z in I,

$$\frac{f(x)+f(y)+f(z)}{3} + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3}\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right] \quad (1)$$

2. If f is concave, then for any three points x, y, z in I,

$$\frac{f(x)+f(y)+f(z)}{3} + f\left(\frac{x+y+z}{3}\right) \leq \frac{2}{3}\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right] \quad (2)$$

When f is strictly convex or concave, the inequality is strict except for $x = y = z$.

Corollary. For a continuous function f, then it is convex if and only if the (1) inequality holds, and it is concave if and only if the (2) inequality holds, $\forall x, y, z \in I$.

General Form. Let f be a function defined on an interval $I \subseteq \mathbb{R}$.

1. If f is convex, then for any $a_1.a_2, ..., a_n \in I$,

$$\sum_{i=1}^n f(a_i) + n(n-2)f\left(\frac{a_1+a_2+...+a_n}{n}\right) \geq (n-1)\sum_{i=1}^n f(b_i)$$

.

2. If f is concave, then for any $a_1.a_2, ..., a_n \in I$,

$$\sum_{i=1}^n f(a_i) + n(n-2)f\left(\frac{a_1+a_2+...+a_n}{n}\right) \leq (n-1)\sum_{i=1}^n f(b_i),$$

with

$$b_i = \frac{1}{n-1}\sum_{j\neq i} a_j, \quad i \in \{\overline{1;n}\},$$

with equality if and only if $a_1 = a_2 = ... = a_n$.

ACZEL'S INEQUALITY

Elementary Form. If $a_1^2 > a_2^2 + ... + a_n^2$ or $b_1^2 > b_2^2 + ... + b_n^2$, then

$$(a_1b_1 - a_2b_2 - ... - a_nb_n)^2 \geq (a_1^2 - a_2^2 - ... - a_n^2)(b_1^2 - b_2^2 - ... - b_n^2),$$

with equality if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$

General Form. Let $p_1, p_2, ..., p_m \geq 1$ such that $\sum_{1\leq i\leq m} 1/p_i = 1$ and let $\{a_{11}, ..., a_{1n}\}, ..., \{a_{m1}, ..., a_{mn}\}$ be m sequences of positive real numbers such that $a_{i1}^{p_i} - a_{i2}^{p_i} - ... - a_{in}^{p_i} > 0$, for $i = \overline{1;m}$. Then,

$$\prod_{i=1}^m a_{i1} - \prod_{i=1}^m a_{i2} - ... - \prod_{i=1}^m a_{in} \geq \prod_{i=1}^m (a_{i1}^{p_i} - a_{i2}^{p_i} - ... - a_{in}^{p_i})^{\frac{1}{p_i}},$$

with equality if and only if all the sequences are proportional.

SURANYI'S INEQUALITY

Elementary Form. For any non-negative numbers $a_1, a_2, ..., a_n$, we have

$$(n-1)\sum_{i=1}^n a_i^n + n\prod_{i=1}^n a_i \geq \left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n a_i^{n-1}\right).$$

General Form. Let $x_k \in I \subseteq \mathbb{R}, k \in \{\overline{1;n}\}$ such that f and f' are convex functions on I, then

$$(n-1)\sum_{k=1}^n f(x_k) + nf\left(\frac{1}{n}\sum_{k=1}^n x_k\right) \geq \sum_{i,j=1}^n f\left(\frac{(n-1)x_i + x_j}{n}\right).$$

By letting $f(x) = e^{nx}$ and $e^{x_k} = a_k, k \in \{\overline{1;n}\}$, we obtain the classical form of the Suranyi's Inequality.

OTHER INEQUALITIES

Vasc's Inequality Let a, b, c be non-negative real numbers. Then,

$$(a^2+b^2+c^2)^2 \geq 3(a^3b + b^3c + c^3a),$$

with equality if and only if $a = b = c$ or $a : b : c = \sin^2(\frac{4\pi}{7}) : \sin^2(\frac{2\pi}{7}) : \sin^2(\frac{2\pi}{7})$ and permutations.

Turkevich Inequality. Let a, b, c, d be non-negative real numbers, then,

$$a^4 + b^4 + c^4 + d^4 + 2abcd \geq a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2 + a^2c^2 + b^2d^2.$$

Shlefier's Inequality. For any real numbers $a_1, a_2, ..., a_n \geq 0$, we have

$$(n-1)\sum_{i=1}^n a_i^4 + n\left(\prod_{i=1}^n a_i\right)^{4/n} \geq \left(\sum_{i=1}^n a_i^2\right).$$

Huygens's Inequality. Let $\{a_i\}_{i=1}^n = \{a_1, a_2, ..., a_n\}$ and $\{b_i\}_{i=1}^n = \{b_1, b_2, ..., b_n\}$ be two sequences (or lists) of positive real numbers. Let $w_1, w_2, ..., w_n$ be positive real numbers such that $w_1 + w_2 + ... + w_n = 1$. Then,

$$\prod_{i=1}^n (a_i + b_i)^{w_i} \geq \prod_{i=1}^n a_i^{w_i} + \prod_{i=1}^n b_i^{w_i}.$$

Mahler's Inequality. Given two sequences of positive real numbers $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$. The inequality states that the geometric mean of the term-by-term sum of two positive sequences $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ is greater or equal to the sum of their two separate geometric means.

$$\prod_{i=1}^n (a_i + b_i)^{1/n} \geq \prod_{i=1}^n a_i^{1/n} + \prod_{i=1}^n b_i^{1/n},$$

where $a_i, b_i > 0, \forall k \in \{\overline{1;n}\}$. As noticed, this is the special case of the Huygen's Inequality

Heinz's Inequality. For real numbers $a, b > 0$ and $\alpha \in [0, 1]$, we have:

$$\sqrt{ab} \leq \frac{a^\alpha b^{1-\alpha} + a^{1-\alpha}b^\alpha}{2} \leq \frac{a+b}{2}.$$

Hilbert's Inequality Consider $a_i, b_j \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$, and $p > 1$, then

$$\sum_{i=1}^n \left(\sum_{i=1}^n \frac{a_ib_j}{i+j}\right) \leq \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^n b_j^q\right)^{\frac{1}{q}}.$$

Weierstrass's Inequality For any real numbers $0 \leq a_1, a_2, ..., a_n \leq 1$, we have

$$(1+a_1)(1+a_2)...(1+a_n) \geq 1 + S_n,$$
$$(1-a_1)(1-a_2)...(1-a_n) \geq 1 - S_n,$$

where $S_n = a_1 + a_2 + ... + a_n$.

LOG-SUM INEQUALITY

Classical Form. Let $a_1, ..., a_n$ and $b_1, ..., b_n$ be non-negative real numbers. Define a and b to be

$$a = \sum_{j=1}^n a_j \quad \text{and} \quad b = \sum_{j=1}^n b_j$$

The Log-Sum inequality states that

$$\sum_{i=1}^n a_i \log\frac{a_i}{b_i} \geq a \cdot \log\frac{a}{b},$$

with equality if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$, or in other words, $\exists k$ such that $a_i = kb_i, \forall i \in \{\overline{1;n}\}$.

Take $a_i \log\frac{a_i}{b_i}$ to be 0 and ∞ if $a_i > 0, b_i = 0$. These are the limiting values obtained as the relevant number tends to 0.

General Form. Let g be a real-valued function whose domain contains $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ such that $g(b_i) > 0 \forall i \in \{\overline{1;n}\}$. Consider another function $f : [m_g, M_g] \to \mathbb{R}$ for which $h(x) = xf(x)$ is convex, where

$$m_g := \min_i\left\{\frac{g(a_i)}{g(b_i)}\right\} \quad \text{and} \quad M_g := \max_i\left\{\frac{g(a_i)}{g(b_i)}\right\}.$$

Then, the general Log-Sum Inequality states that:

$$\sum_{i=1}^n g(a_i) f\left(\frac{g(a_i)}{g(b_i)}\right) \geq \left(\sum_{i=1}^n g(a_i)\right) f\left(\frac{\sum_{i=1}^n g(a_i)}{\sum_{i=1}^n g(b_i)}\right)$$

Remark. Define $g(x) = x^r, r \in \mathbb{R}$, and $f(x) = \log(x), x > 0$. As $h(x) = x\log(x)$ is a convex function, therefore, we obtain

$$\sum_{i=1}^n a_i^r \log\frac{a_i}{b_i} \geq \left(\sum_{i=1}^n a_i^r\right)\log\left(\frac{\sum_{i=1}^n a_i^r}{\sum_{i=1}^n b_i^r}\right)$$

By substituting $r = 1$, we obtain the classical form of the Log-Sum inequality.