Table of Derivatives [1]

FUNCTION OF ONE-VARIABLE

Definition. The derivative $f'(x)$ of $f(x)$ is defined by
$$f'(x) = \frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$
If the limit exists, then f is said to be differentiable at x.

Lagrange's Mean Value Theorem. If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists a point $c \in (a,b)$ such that
$$f(b) - f(a) = (b-a)f'(c).$$
If $f'(x) \geq 0\ (\leq 0)\ (=0)$ for all $x \in (a,b)$, then f is a constant function on (a,b).

Cauchy's Mean Value Theorem. If $f(x)$ and $g(x)$ are continuous functions on $[a,b]$, differentiable on (a,b), and $g'(x) \neq 0,\ \forall x \in (a,b)$. Then there is a point $c \in (a,b)$ such that
$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Linearity, Product, and Quotient Rules. Let $u = f(x)$, $v = g(x)$, and α, β be constants. Then,

1. $\dfrac{d}{dx}(\alpha u + \beta v) = \alpha u' + \beta v'$
2. $\dfrac{d}{dx}(u \cdot v) = u'v + v'u$
3. $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{u'v - v'u}{v^2} \quad \Rightarrow \quad \dfrac{d}{dx}\left(\dfrac{1}{v}\right) = -\dfrac{v'}{v^2}$

For a collection of functions $f_1, f_2, ..., f_n$,
$$\frac{d}{dx}(f_1 f_2 ... f_n)(x) = (f_1' f_2 ... f_n)(x) + (f_1 f_2' ... f_n)(x) + ... + (f_1 f_2 ... f_n')(x)$$

The Chain Rule. If $u = g(x)$ is differentiable at x and f is differentiable at u. Then,
$$\frac{d}{dx} f(u) = u' \cdot f'(u), \quad \text{or} \quad \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

The Inverse Function Rule. Let f be a bijective function with inverse $f^{-1}(x)$, and f is differentiable at $f^{-1}(x)$. Then,
$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Basic Power Rules.

1. $\dfrac{d}{dx}(c) = 0$
2. $\dfrac{d}{dx}x = 1$
3. $\dfrac{d}{dx}\sqrt{x} = \dfrac{1}{2\sqrt{x}}$
4. $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$
5. $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x}}\right) = -\dfrac{1}{2\sqrt{x^3}}$
6. $\dfrac{d}{dx}[cf(x)] = c \cdot f'(x)$
7. $\dfrac{d}{dx}x^n = nx^{n-1},\ \forall n \in \mathbb{R}$
8. $\dfrac{d}{dx}\sqrt[n]{x} = \dfrac{1}{n\sqrt[n]{x^{n-1}}}$
9. $\dfrac{d}{dx}\left(\dfrac{1}{x^n}\right) = -\dfrac{n}{x^{n+1}}$
10. $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt[n]{x}}\right) = -\dfrac{1}{n\sqrt[n]{x^{n+1}}}$

Trigonometric Functions.

1. $\dfrac{d}{dx}\sin x = \cos x$
2. $\dfrac{d}{dx}\cos x = -\sin x$
3. $\dfrac{d}{dx}\tan x = \sec^2 x$
   $= 1 + \tan^2 x$
4. $\dfrac{d}{dx}\csc x = -\csc x \cot x$
5. $\dfrac{d}{dx}\sec x = \sec x \tan x$
6. $\dfrac{d}{dx}\cot x = -\csc^2 x$
   $= -(1 + \cot^2 x)$

Inverse Trigonometric Functions.

1. $\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$
2. $\dfrac{d}{dx}\cos^{-1} x = -\dfrac{1}{\sqrt{1-x^2}}$
3. $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1+x^2}$
4. $\dfrac{d}{dx}\csc^{-1} x = -\dfrac{1}{|x|\sqrt{x^2-1}}$ $(*)$
5. $\dfrac{d}{dx}\sec^{-1} x = \dfrac{1}{|x|\sqrt{x^2-1}}$ $(*)$
6. $\dfrac{d}{dx}\cot^{-1} x = -\dfrac{1}{1+x^2}$

$(*)$ The range of $\csc^{-1} x$ and $\sec^{-1} x$ in 4. and 5. is, respectively,
$$f(D) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \backslash \{0\} \quad \text{and} \quad f(D) = [0,\pi] \backslash \left\{\frac{\pi}{2}\right\}$$
Some authors define the range of $\csc^{-1} x$ and $\sec^{-1} x$ to be, respectively,
$$\left(-\pi, -\frac{\pi}{2}\right] \cup \left(0, \frac{\pi}{2}\right] \quad \text{and} \quad \left[0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right].$$

- If $f(D(\csc^{-1} x)) = \left(-\pi, -\frac{\pi}{2}\right] \cup \left(0, \frac{\pi}{2}\right]$, then $\dfrac{d}{dx}\csc^{-1} x = \dfrac{-1}{x\sqrt{x^2-1}}$
- If $f(D(\sec^{-1} x)) = \left[0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right]$, then $\dfrac{d}{dx}\sec^{-1} x = \dfrac{1}{x\sqrt{x^2-1}}$

Hyperbolic Functions.

1. $\dfrac{d}{dx}\sinh x = \cosh x$
2. $\dfrac{d}{dx}\cosh x = \sinh x$
3. $\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$
   $= 1 - \tanh^2 x$
4. $\dfrac{d}{dx}\operatorname{csch} x = -\coth x \operatorname{csch} x$
5. $\dfrac{d}{dx}\operatorname{sech} x = -\tanh x \operatorname{sech} x$
6. $\dfrac{d}{dx}\coth x = -\operatorname{csch}^2 x$
   $= 1 - \coth^2 x$

Inverse Hyperbolic Functions.

1. $\dfrac{d}{dx}\sinh^{-1} x = \dfrac{1}{\sqrt{1+x^2}}$
2. $\dfrac{d}{dx}\cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}}$
3. $\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1-x^2}$,
   $D = (-1, 1)$
4. $\dfrac{d}{dx}\operatorname{csch}^{-1} x = -\dfrac{1}{|x|\sqrt{x^2+1}}$
5. $\dfrac{d}{dx}\operatorname{sech}^{-1} x = -\dfrac{1}{x\sqrt{1-x^2}}$
6. $\dfrac{d}{dx}\coth^{-1} x = \dfrac{1}{1-x^2}$,
   $D = (-\infty, -1) \cup (1, +\infty)$

Exponential and Logarithms.

1. $\dfrac{d}{dx}e^x = e^x$
2. $\dfrac{d}{dx}a^x = a^x \ln a,\ a > 0$
3. $\dfrac{d}{dx}\ln x = \dfrac{1}{x},\ x > 0$
4. $\dfrac{d}{dx}\ln |x| = \dfrac{1}{x},\ x \neq 0$
5. $\dfrac{d}{dx}\log_b x = \dfrac{1}{x \ln b}$,
   $x > 0,\ b > 0$
6. $\dfrac{d}{dx}\log_b |x| = \dfrac{1}{x \ln b}$,
   $x \neq 0,\ b > 0$

Other Rules.

1. $\dfrac{d}{dx}|x| = \dfrac{x}{|x|}$
2. $\dfrac{d}{dx}x^x = x^x(1 + \ln x)$
3. $\dfrac{d}{dx}|f(x)| = \dfrac{f(x)}{|f(x)|} \cdot f'(x)$
4. $\dfrac{d}{dx}u^v = u^v \cdot \left(\dfrac{u'}{u} \cdot v + v' \ln u\right)$

Quotient of Two Polynomials. Let $p(x) = a_0 + a_1 x + ... + a_n x^n$ and $h(x) = b_0 + b_1 x + ... + b_n x^n$ be two polynomials of degree n. Then,
$$\frac{d}{dx}\left(\frac{p(x)}{h(x)}\right) = \frac{1}{h^2(x)} \sum_{k=1}^{n} \left(\sum_{i=0}^{k-1} (k-i) \begin{vmatrix} a_k & a_i \\ b_k & b_i \end{vmatrix} x^{i+k-1} \right),$$
where $| :: |$ is the determinant, $h \neq 0$, and the total number of unique determinants is $n(n+1)/2$. In particular, for $n = 1$ and $n = 2$,

1. $\dfrac{d}{dx}\left(\dfrac{a_1 x + a_0}{b_1 x + b_0}\right) = \dfrac{\begin{vmatrix} a_1 & a_0 \\ b_1 & b_0 \end{vmatrix}}{(b_1 x + b_0)^2}$
2. $\dfrac{d}{dx}\left(\dfrac{a_2 x^2 + a_1 x + a_0}{b_2 x^2 + b_1 x + b_0}\right) = \dfrac{\begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} x^2 + 2\begin{vmatrix} a_2 & a_0 \\ b_2 & b_0 \end{vmatrix} x + \begin{vmatrix} a_1 & a_0 \\ b_1 & b_0 \end{vmatrix}}{(b_2 x^2 + b_1 x + b_0)^2}$

As observed, the number of unique determinants of 1. is $1(1+1)/2 = 1$, and of 2. is $2(2+1)/2 = 3$.

Note: the degrees of two polynomials don't need to be equal. Set some of the coefficients to zero and the formula still works.

[1] This note was written by Don D. Le.

L'Hôpital's Rule. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ (or ∞) and f, g are differentiable functions on an interval I, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided that the limit on the right-hand side exists.

Derivative of Parametric Equations. Given a parametric equation $(x, y) = (f(t), g(t))$, $t \in I$, where f and g are differentiable functions and $f' \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}.$$

If f and g are two times differentiable, then

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{x'y'' - x''y}{(x')^3}$$

Derivative of Polar Equations. Given a polar curve $r = f(\theta)$ defined by the parametric equations $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$, then

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}.$$

Higher Derivatives.

1. $\dfrac{d^n}{dx^n}x^m = m(m-1)\cdots(m-n+1)x^{m-n}$
2. $\dfrac{d^n}{dx^n}\sqrt{x} = (-1)^{n-1}\dfrac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^{-(n-\frac{1}{2})}$
3. $\dfrac{d^n}{dx^n}\left(\dfrac{1}{x}\right) = (-1)^n\dfrac{n!}{x^n}$
4. $\dfrac{d^n}{dx^n}e^{\alpha x} = \alpha^n e^{\alpha x}$
5. $\dfrac{d^n}{dx^n}a^{\beta x} = \beta^n(\ln\beta)^n a^x$
6. $\dfrac{d^n}{dx^n}\ln x = (-1)^{n-1}\dfrac{(n-1)!}{x^n}$
7. $\dfrac{d^n}{dx^n}\log_a x = (-1)^{n-1}\dfrac{(n-1)!}{x^n\ln a}$
8. $\dfrac{d^n}{dx^n}\sin x = \sin\left(x + \dfrac{n\pi}{2}\right) = \begin{cases}\sin x & \text{if } n \equiv 0 \pmod 4\\ \cos x & \text{if } n \equiv 1 \pmod 4\\ -\sin x & \text{if } n \equiv 2 \pmod 4\\ -\cos x & \text{if } n \equiv 3 \pmod 4\end{cases}$
9. $\dfrac{d^n}{dx^n}\cos x = \cos\left(x + \dfrac{n\pi}{2}\right) = \begin{cases}\cos x & \text{if } n \equiv 0 \pmod 4\\ -\sin x & \text{if } n \equiv 1 \pmod 4\\ -\cos x & \text{if } n \equiv 2 \pmod 4\\ \sin x & \text{if } n \equiv 3 \pmod 4\end{cases}$
10. $\dfrac{d^n}{dx^n}\sin ax = a^n\sin\left(ax + \dfrac{n\pi}{2}\right)$
11. $\dfrac{d^n}{dx^n}\cos ax = a^n\cos\left(ax + \dfrac{n\pi}{2}\right)$
12. $\dfrac{d^n}{dx^n}\sin^2 x = -2^{n-1}\sin\left(2x + \dfrac{n\pi}{2}\right)$
13. $y = \tan^{-1}x,\ \dfrac{d^n y}{dx^n} = (n-1)!\cos^n y\sin\left(ny + \dfrac{n\pi}{2}\right)$
14. $y = \cot^{-1}x,\ \dfrac{d^n y}{dx^n} = (-1)^n(n-1)!\sin^n y\sin ny$
15. $y = e^{ax}\sin bx,\ \dfrac{d^n y}{dx^n} = (a^2+b^2)^{n/2}e^{ax}\sin\left(bx + n\tan^{-1}\dfrac{b}{a}\right)$
16. $y = e^{ax}\cos bx,\ \dfrac{d^n y}{dx^n} = (a^2+b^2)^{n/2}e^{ax}\cos\left(bx + n\tan^{-1}\dfrac{b}{a}\right)$
17. $\dfrac{d^n}{dx^n}\sinh x = \begin{cases}\sinh x & \text{if } n \text{ is even}\\ \cosh x & \text{if } n \text{ is odd}\end{cases}$
18. $\dfrac{d^n}{dx^n}\cosh x = \begin{cases}\cosh x & \text{if } n \text{ is even}\\ \sinh x & \text{if } n \text{ is odd}\end{cases}$

Leibniz Rule. If f and g are n-times differentiable, then

$$\frac{d^n}{dx^n}[f(x)g(x)] = \sum_{k=0}^{n}\binom{n}{k}f^{(n-k)}(x)g^{(k)}(x)$$

Faà di Bruno Formula. If f and g are n-times differentiable functions, then

$$\frac{d^n}{dx^n}[f(g(x))] = n!\sum_{\{k_m\}}f^{(r)}(g(x))\prod_{m=1}^{n}\frac{1}{k_m!}\left(g^{(m)}(x)\right)^{k_m}$$

where $r = \sum_{m=1}^{n-1}k_m$ and the set $\{k_m\}$ consists of all non-negative integer solution of the Diophantine equation $\sum_{m=1}^{n}mk_m = n$.

FUNCTION OF TWO OR MORE VARIABLES

Partial Derivative. The first partial derivatives of a function $f(x, y)$ is defined by

$$\frac{\partial f}{\partial x} = D_x f = f_x = \lim_{h\to 0}\frac{f(x+h, y) - f(x, y)}{h},$$
$$\frac{\partial f}{\partial y} = D_y f = f_y = \lim_{h\to 0}\frac{f(x, y+h) - f(x, y)}{h}.$$

In general, let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a function of n-variables. Then, the partial derivative of f is defined by

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0}\frac{f(\mathbf{x} + h\mathbf{e_i}) - f(\mathbf{x})}{h}$$

where $\mathbf{e_i}$ is the i^{th} row (or column) of the identity matrix I_n.

Theorem 1 (**Clairaut's Theorem**)**.** Suppose $f : U \in \mathbb{R}^2 \to \mathbb{R}$ have all partial derivatives up to second order in a region containing $p = (a, b)$, then

$$\frac{\partial^2 f}{\partial x\partial y}(p) = \frac{\partial^2 f}{\partial y\partial x}(p).$$

The Chain Rule.

1. $\dfrac{d}{dt}f(x(t), y(t)) = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} = f_x x' + f_y y'$
2. $\dfrac{\partial}{\partial u}f(x(u,v), y(u,v)) = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u} = f_x x_u + f_y y_u$
3. $\dfrac{\partial}{\partial v}f(x(u,v), y(u,v)) = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v} = f_x x_v + f_y y_v$

The Generalized Chain Rule.
Suppose that $f(\mathbf{x}) = (x_1, x_2, ..., x_n)$ is a differentiable functions of n-variables, and $x_k(\mathbf{t}) = (t_1, t_2, ..., t_m)$, $1 \le k \le n$ is a differentiable function of m-variables. Then,

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i},$$

for each $i = 1, 2, ..., m$.

Implicit Differentiation. If $F(x, y) = 0$, then the derivative of the implicit function $y = y(x)$ is given by

$$y' = \frac{dy}{dx} = -\frac{F_x}{F_y}$$

Similarly for $F(x, y, z) = 0$, the derivatives of the implicit function $z = z(x, y)$ is given by

$$z_x = -\frac{F_x}{F_z}, \quad \text{and} \quad z_y = -\frac{F_y}{F_z}$$

Directional Derivative. Suppose f is a function defined on a neighborhood $N_\varepsilon(x_0, y_0)$ and let $\mathbf{v} = \langle v_1, v_2\rangle$ be a unit vector. Then, the directional derivative of f in the direction of $\mathbf{v}$ is defined by

$$D_{\mathbf{v}}f(x_0, y_0) = \lim_{h\to 0}\frac{f(x_0 + hv_1, y_0 + hv_2) - f(x_0, y_0)}{h},$$

provided that the limit exists. If $v = \langle 1, 0\rangle$, or $v = \langle 0, 1\rangle$, then this is just the regular partial derivatives of f at (x_0, y_0).

In general, the directional derivative of the function $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ in the direction of the unit vector $\mathbf{v} = (v_1, v_2, ..., v_n)$ is defined by

$$D_{\mathbf{v}}f(\mathbf{x}) = \lim_{h\to 0}\frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h}.$$

Theorem 2. If f is differentiable at $\mathbf{x}$, then f has a directional derivative in all directions of the unit vector $\mathbf{v}$, and one has

$$D_{\mathbf{v}}f(\mathbf{x}) = \langle f_{x_1}, ..., f_{x_n}\rangle\cdot\langle v_1, ..., v_n\rangle = \nabla f(\mathbf{x})\cdot\mathbf{v}.$$

Theorem 3. Suppose f is a differentiable function of n-variables.

(i) The maximum value of $D_{\mathbf{v}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{v}$ has the same direction as $\nabla f(\mathbf{x})$.

(ii) Similarly, the minimum value of $D_{\mathbf{v}}f(\mathbf{x})$ is $-|\nabla f(\mathbf{x})|$, and it occurs when $\mathbf{v}$ has the opposite direction as $\nabla f(\mathbf{x})$.

(iii) If $|\nabla f(\mathbf{x})| = 0$, then $D_{\mathbf{v}}f(\mathbf{x}) = 0$.