Partial Differential Equations

INTRODUCTION AND CLASSIFICATION

1 Basic Definitions

PDE. A partial differential equation (PDE) is an equation involving an unknown function u of two or more independent variables and its partial derivatives:

$$F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_1}, \ldots) = 0.$$

Order. The order of a PDE is the order of the highest partial derivative appearing in the equation.

Linear PDE. A PDE is linear if it is linear in u and all its derivatives. A general second-order linear PDE in two variables is

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

where A, B, C, D, E, F, G are functions of x and y. The PDE is homogeneous if $G \equiv 0$.

Quasilinear PDE. A PDE is quasilinear if it is linear in the highest-order derivatives. For example,

$$A(x, y, u, u_x, u_y)u_{xx} + 2B(\cdots)u_{xy} + C(\cdots)u_{yy} = D(\cdots)$$

is quasilinear but not necessarily linear (since A, B, C, D may depend on u, u_x, u_y).

Superposition Principle. If u_1 and u_2 are solutions of a linear homogeneous PDE, then $c_1u_1 + c_2u_2$ is also a solution for any constants c_1, c_2.

Examples of Common PDEs.

1. **Heat equation:** $u_t = ku_{xx}$ (parabolic, 2nd order, linear, homogeneous).
2. **Wave equation:** $u_{tt} = c^2u_{xx}$ (hyperbolic, 2nd order, linear, homogeneous).
3. **Laplace's equation:** $u_{xx} + u_{yy} = 0$ (elliptic, 2nd order, linear, homogeneous).
4. **Poisson's equation:** $u_{xx}+u_{yy} = f(x, y)$ (elliptic, 2nd order, linear, non-homogeneous).
5. **Burgers' equation:** $u_t + uu_x = \nu u_{xx}$ (parabolic, 2nd order, quasilinear).
6. **Korteweg-de Vries:** $u_t + 6uu_x + u_{xxx} = 0$ (3rd order, quasilinear).
7. **Schrödinger:** $iu_t + u_{xx} + V(x)u = 0$ (parabolic-type, linear).

2 Classification of Second-Order Linear PDEs

Consider the second-order PDE

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + (\text{lower order terms}) = 0.$$

The discriminant is $\Delta = B^2 - AC$. The PDE is classified as:

1. **Elliptic** if $\Delta < 0$ (e.g., Laplace's equation $u_{xx} + u_{yy} = 0$).
2. **Parabolic** if $\Delta = 0$ (e.g., heat equation $u_t = ku_{xx}$).
3. **Hyperbolic** if $\Delta > 0$ (e.g., wave equation $u_{tt} = c^2u_{xx}$).

Remark. The classification may vary from point to point if A, B, C depend on x, y. For example, the Tricomi equation $u_{xx} + yu_{yy} = 0$ is elliptic for $y > 0$, parabolic for $y = 0$, and hyperbolic for $y < 0$.

Canonical Forms. By an appropriate change of variables (ξ, η), a second-order PDE can be transformed into its canonical form:

- Elliptic: $u_{\xi\xi} + u_{\eta\eta} = (\text{lower order terms})$.
- Parabolic: $u_{\xi\xi} = (\text{lower order terms})$.
- Hyperbolic: $u_{\xi\eta} = (\text{lower order terms})$.

Characteristic Curves. The characteristic curves of $Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0$ satisfy

$$A\left(\frac{dy}{dx}\right)^2 - 2B\left(\frac{dy}{dx}\right) + C = 0,$$

giving $\dfrac{dy}{dx} = \dfrac{B \pm \sqrt{B^2 - AC}}{A}$. Hyperbolic equations have two families of real characteristics, parabolic equations have one, and elliptic equations have none (in $\mathbb{R}$).

Example. Classify and find the canonical form of $u_{xx} - 4u_{xy} + 4u_{yy} = 0$.
Solution. Here $A = 1$, $B = -2$, $C = 4$. The discriminant is $\Delta = (-2)^2 - 1 \cdot 4 = 0$, so the PDE is **parabolic**. The characteristic equation is

$$\frac{dy}{dx} = \frac{-2 \pm 0}{1} = -2 \quad \Rightarrow \quad y = -2x + c.$$

Let $\xi = y + 2x$ and $\eta = x$ (any independent second variable). Under this change of variables, the PDE transforms to

$$u_{\eta\eta} = 0 \quad \Rightarrow \quad u = \eta f(\xi) + g(\xi) = xf(y + 2x) + g(y + 2x).$$

Example. Classify $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$.
Solution. $A = 3$, $B = 5$, $C = 3$. $\Delta = 25 - 9 = 16 > 0$, so it is **hyperbolic**. The characteristic slopes are $\dfrac{dy}{dx} = \dfrac{5 \pm 4}{3}$, giving $y = 3x + c_1$ and $y = \frac{1}{3}x + c_2$. Let $\xi = y - 3x$, $\eta = y - x/3$. The canonical form is $u_{\xi\eta} = 0$, with general solution $u = F(y - 3x) + G(y - x/3)$.

3 Well-Posedness (Hadamard)

A PDE problem is **well-posed** in the sense of Hadamard if:

1. A solution exists.
2. The solution is unique.
3. The solution depends continuously on the data.

If any of these conditions fails, the problem is **ill-posed**.

Types of Conditions.

- **Initial conditions** (Cauchy data): prescribed u and/or u_t at $t = 0$.
- **Boundary conditions**: Dirichlet (u prescribed), Neumann (u_n prescribed), Robin ($\alpha u + \beta u_n$ prescribed), or periodic.

FIRST-ORDER PDEs

4 Transport Equation

The simplest first-order PDE is the transport equation

$$u_t + cu_x = 0, \quad c \in \mathbb{R}.$$

The general solution is $u(x, t) = f(x - ct)$, where f is determined by the initial condition $u(x, 0) = f(x)$. The solution represents a wave traveling to the right with speed c.

Non-homogeneous Transport. For $u_t + cu_x = g(x, t)$ with $u(x, 0) = f(x)$, the solution is

$$u(x, t) = f(x - ct) + \int_0^t g(x - c(t - \tau), \tau)\, d\tau.$$

Example. Solve $u_t + 3u_x = 0$, $u(x, 0) = e^{-x^2}$.
Solution. The solution is $u(x, t) = e^{-(x-3t)^2}$. This is a Gaussian pulse traveling to the right at speed 3.

Example. Solve $u_t + 2u_x = x$, $u(x, 0) = \sin x$.
Solution. Characteristics: $x = x_0 + 2t$, so $x_0 = x - 2t$. Along a characteristic, $\dfrac{du}{dt} = x_0 + 2t$, so

$$u = \sin(x_0) + \int_0^t (x_0 + 2\tau)d\tau = \sin(x - 2t) + (x - 2t)t + t^2.$$

5 Method of Characteristics

Linear Case. Consider the first-order linear PDE

$$a(x, y)u_x + b(x, y)u_y = c(x, y)u + d(x, y).$$

The characteristic equations are the ODE system

$$\frac{dx}{ds} = a(x, y), \quad \frac{dy}{ds} = b(x, y), \quad \frac{du}{ds} = c(x, y)u + d(x, y).$$

The first two equations define the characteristic curves in the xy-plane. Along these curves, the PDE reduces to an ODE for u.

Quasilinear Case. For the quasilinear PDE

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u),$$

the characteristic equations are

$$\frac{dx}{ds} = a(x, y, u), \quad \frac{dy}{ds} = b(x, y, u), \quad \frac{du}{ds} = c(x, y, u).$$

Example. Solve $u_t + uu_x = 0$, $u(x, 0) = f(x)$ (inviscid Burgers' equation).
Solution. The characteristic equations are

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = u, \quad \frac{du}{ds} = 0.$$

From $du/ds = 0$, u is constant along characteristics. From $dt/ds = 1$, we have $t = s$. From $dx/ds = u = f(x_0)$, where x_0 is the initial position, we get $x = x_0 + f(x_0)t$. Therefore, the implicit solution is

$$u(x, t) = f(x - ut).$$

This solution may develop shocks (discontinuities) when characteristics cross.

Example. Solve $xu_x + yu_y = u$, $u(x,1) = g(x)$.
Solution. The characteristic equations are

$$\frac{dx}{ds} = x, \quad \frac{dy}{ds} = y, \quad \frac{du}{ds} = u.$$

Solving: $x = x_0 e^s$, $y = y_0 e^s$, $u = u_0 e^s$. From the initial curve $y_0 = 1$, so $y = e^s$ and $s = \ln y$. Then $x = x_0 y$ and $u = u_0 y = g(x_0)y = g(x/y) \cdot y$. Therefore,

$$u(x,y) = yg(x/y).$$

Example. Solve $u_x + 2u_y = u^2$, $u(x,0) = x$.
Solution. Characteristic equations: $\frac{dx}{ds} = 1$, $\frac{dy}{ds} = 2$, $\frac{du}{ds} = u^2$. From the first two: $x = s + x_0$, $y = 2s$, so $s = y/2$ and $x_0 = x - y/2$. The u-equation is a Bernoulli ODE:

$$\frac{du}{ds} = u^2 \quad \Rightarrow \quad -\frac{1}{u} = -s + C \quad \Rightarrow \quad u = \frac{1}{1/u_0 - s}$$

at $s = 0$: $u_0 = x_0 = x - y/2$. Thus,

$$u(x,y) = \frac{x - y/2}{1 - (x - y/2)y/2} = \frac{2x - y}{2 - (2x - y)y/2}.$$

The solution blows up when $2 - (2x - y)y/2 = 0$.

Example. Solve $u_t + u_x = 0$, $u(x,0) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$.
Solution. Characteristics are $x = t + x_0$. Along each, u is constant. Therefore, $u(x,t) = 1$ if $x - t < 0$ and $u(x,t) = 0$ if $x - t > 0$, i.e.,

$$u(x,t) = \begin{cases} 1, & x < t \\ 0, & x > t \end{cases}$$

The discontinuity travels along $x = t$ at speed 1.

6 Shocks and Weak Solutions

Conservation Law Form. The quasilinear PDE $u_t + f(u)_x = 0$ is called a conservation law with flux $f(u)$.

Rankine-Hugoniot Condition. If a shock (discontinuity) propagates along a curve $x = s(t)$, then the shock speed satisfies

$$\frac{ds}{dt} = \frac{f(u_R) - f(u_L)}{u_R - u_L},$$

where u_L and u_R are the values of u on the left and right of the shock, respectively.

Entropy Condition (Lax). A shock connecting u_L to u_R is admissible if

$$f'(u_L) > \frac{ds}{dt} > f'(u_R).$$

This ensures that characteristics flow into the shock, not out of it.

Rarefaction Waves. When characteristics diverge (rather than collide), the solution develops a rarefaction fan. For $u_t + f(u)_x = 0$ with $f''(u) > 0$ and $u_L < u_R$, the rarefaction wave is

$$u(x,t) = \begin{cases} u_L, & x/t < f'(u_L) \\ (f')^{-1}(x/t), & f'(u_L) \le x/t \le f'(u_R) \\ u_R, & x/t > f'(u_R) \end{cases}$$

Example. Solve $u_t + uu_x = 0$ with $u(x,0) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$.
Solution. Here $f(u) = u^2/2$, so $f'(u) = u$. Since $u_L = 1 > 0 = u_R$, characteristics collide and a shock forms immediately. The Rankine-Hugoniot condition gives

$$\frac{ds}{dt} = \frac{f(u_R) - f(u_L)}{u_R - u_L} = \frac{0 - 1/2}{0 - 1} = \frac{1}{2}.$$

The shock path is $x = t/2$. The solution is

$$u(x,t) = \begin{cases} 1, & x < t/2 \\ 0, & x > t/2 \end{cases}$$

The Lax entropy condition $f'(u_L) = 1 > 1/2 > 0 = f'(u_R)$ is satisfied.

Example (Rarefaction). Solve $u_t + uu_x = 0$ with $u(x,0) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$.
Solution. Now $u_L = 0 < 1 = u_R$ and $f'' = 1 > 0$, so characteristics diverge. A shock would violate the entropy condition. The rarefaction solution is

$$u(x,t) = \begin{cases} 0, & x/t < 0 \\ x/t, & 0 \le x/t \le 1 \\ 1, & x/t > 1 \end{cases}$$

THE HEAT EQUATION

7 Derivation

The one-dimensional heat equation models the diffusion of heat in a rod:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \ t > 0,$$

where $u(x,t)$ is the temperature, and $k > 0$ is the thermal diffusivity. This is derived from Fourier's law of heat conduction $q = -K_0 u_x$ and conservation of energy.

8 Separation of Variables (Dirichlet BCs)

Problem. Solve

$$u_t = ku_{xx}, \quad u(0,t) = 0, \ u(L,t) = 0, \ u(x,0) = f(x).$$

Solution. Let $u(x,t) = X(x)T(t)$. Substituting:

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda \quad \text{(separation constant)}.$$

This gives the eigenvalue problem $X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$, and $T' + k\lambda T = 0$.

The eigenvalues and eigenfunctions are

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

The time part is $T_n(t) = e^{-k\lambda_n t}$. By superposition:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-kn^2\pi^2 t/L^2},$$

where $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$.

Example. Solve $u_t = u_{xx}$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = \sin x + 3\sin 2x$.
Solution. Here $k = 1$, $L = \pi$, so $\lambda_n = n^2$. The initial condition is already in the eigenfunction basis: $B_1 = 1$, $B_2 = 3$, all other $B_n = 0$. Therefore,

$$u(x,t) = e^{-t}\sin x + 3e^{-4t}\sin 2x.$$

The $\sin 2x$ mode decays 4 times faster than the $\sin x$ mode. As $t \to \infty$, the solution is dominated by the slowest-decaying mode: $u \approx e^{-t}\sin x$.

Example. Solve $u_t = u_{xx}$, $u(0,t) = u(1,t) = 0$, $u(x,0) = x(1-x)$.
Solution. Here $L = 1$. Compute B_n:

$$B_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx = \frac{4}{n^3\pi^3}[1 - (-1)^n] = \begin{cases} 8/(n^3\pi^3), & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Therefore,

$$u(x,t) = \frac{8}{\pi^3}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^3}\sin(n\pi x)\, e^{-n^2\pi^2 t}.$$

9 Neumann Boundary Conditions (Insulated Ends)

Problem. Solve $u_t = ku_{xx}$, $u_x(0,t) = u_x(L,t) = 0$, $u(x,0) = f(x)$.

Solution. The eigenvalue problem is $X'' + \lambda X = 0$, $X'(0) = X'(L) = 0$. The eigenvalues and eigenfunctions are

$$\lambda_0 = 0, \ X_0 = 1; \quad \lambda_n = \frac{n^2\pi^2}{L^2}, \ X_n(x) = \cos\frac{n\pi x}{L}, \quad n = 1, 2, \ldots$$

The solution is

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-kn^2\pi^2 t/L^2},$$

where $A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$.

Remark. The $A_0/2$ term represents the steady-state temperature (the average of the initial condition).

Example. Solve $u_t = u_{xx}$, $u_x(0,t) = u_x(\pi,t) = 0$, $u(x,0) = \cos^2 x$.
Solution. Here $L = \pi$. Using $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$, the initial condition is already in the cosine basis: $A_0/2 = 1/2$ and $A_2 = 1/2$, all others zero. Therefore,

$$u(x,t) = \frac{1}{2} + \frac{1}{2}e^{-4t}\cos 2x.$$

As $t \to \infty$, $u \to 1/2$, which is the average of $\cos^2 x$ over $[0,\pi]$.

10 Robin Boundary Conditions

Problem. Solve $u_t = ku_{xx}$ with $u_x(0,t) - h_1 u(0,t) = 0$ and $u_x(L,t) + h_2 u(L,t) = 0$, where $h_1, h_2 > 0$ (radiation/convection at the ends).

The eigenvalue problem $X'' + \lambda X = 0$ with these BCs leads to eigenvalues λ_n determined by a transcendental equation (no closed form in general). The eigenfunctions are

$$X_n(x) = \cos(\sqrt{\lambda_n}\, x) + \frac{h_1}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n}\, x).$$

11 Non-Homogeneous Heat Equation

Problem. Solve

$$u_t = ku_{xx} + Q(x,t), \quad u(0,t) = 0,\ u(L,t) = 0,\ u(x,0) = f(x).$$

Eigenfunction Expansion. Expand $u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L}$ and $Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin \frac{n\pi x}{L}$, where

$$q_n(t) = \frac{2}{L} \int_0^L Q(x,t) \sin \frac{n\pi x}{L}\, dx.$$

Substituting gives the ODE $b_n' + k\lambda_n b_n = q_n(t)$, with solution

$$b_n(t) = B_n e^{-k\lambda_n t} + e^{-k\lambda_n t} \int_0^t q_n(\tau) e^{k\lambda_n \tau}\, d\tau,$$

where $B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$.

12 Non-Homogeneous Boundary Conditions

Problem. Solve $u_t = ku_{xx}$ with $u(0,t) = \alpha(t)$, $u(L,t) = \beta(t)$, $u(x,0) = f(x)$.

Strategy. Let $u(x,t) = v(x,t) + w(x,t)$, where w is chosen to satisfy the BCs:

$$w(x,t) = \alpha(t) + \frac{x}{L}[\beta(t) - \alpha(t)].$$

Then v satisfies the heat equation with homogeneous BCs:

$$v_t = kv_{xx} - w_t, \quad v(0,t) = 0,\ v(L,t) = 0,\ v(x,0) = f(x) - w(x,0).$$

This is a non-homogeneous heat equation with homogeneous BCs, solvable by eigenfunction expansion.

Example. Solve $u_t = u_{xx}$, $u(0,t) = 0$, $u(1,t) = 1$, $u(x,0) = 0$.
Solution. Let $w(x,t) = x$ (satisfies the BCs). Set $v = u - w$. Then $v_t = v_{xx} - w_t = v_{xx}$, $v(0,t) = v(1,t) = 0$, $v(x,0) = -x$. Expand:

$$B_n = 2\int_0^1 (-x)\sin(n\pi x)\, dx = \frac{2(-1)^n}{n\pi}$$

Thus,

$$u(x,t) = x + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} \sin(n\pi x)\, e^{-n^2\pi^2 t}.$$

As $t \to \infty$, $u \to x$ (the steady-state solution satisfying $u_{xx} = 0$, $u(0) = 0$, $u(1) = 1$).

Steady-State Solutions. For time-independent BCs, the steady-state $u_s(x)$ satisfies $u_s'' = 0$ with the given BCs. On $[0, L]$ with $u_s(0) = \alpha$ and $u_s(L) = \beta$:

$$u_s(x) = \alpha + \frac{\beta - \alpha}{L} x.$$

13 Maximum Principle

Theorem 1 (**Weak Maximum Principle**). Let u satisfy $u_t - ku_{xx} \le 0$ in the rectangle $R = (0,L) \times (0,T]$. Then the maximum of u over $\overline{R}$ is attained on the **parabolic boundary** $\Gamma = \{t=0\} \cup \{x=0\} \cup \{x=L\}$:

$$\max_{\overline{R}} u = \max_{\Gamma} u.$$

Consequences.

1. **Uniqueness:** If u_1 and u_2 both solve the heat equation with the same BCs and IC, then $w = u_1 - u_2$ satisfies $w_t = kw_{xx}$ with zero BCs and IC. By the maximum principle, $w \equiv 0$.
2. **Continuous dependence:** If $|f_1(x) - f_2(x)| \le \varepsilon$ for all x, then $|u_1(x,t) - u_2(x,t)| \le \varepsilon$ for all x, t.
3. **Stability:** Small changes in the initial data produce small changes in the solution.

14 Heat Equation on Infinite Domains

Problem. Solve $u_t = ku_{xx}$, $-\infty < x < \infty$, $t > 0$, $u(x,0) = f(x)$.

Fundamental Solution (Heat Kernel).

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right), \quad t > 0.$$

The solution is a convolution with the initial data:

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t) f(y)\, dy = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)} f(y)\, dy.$$

Properties of the Heat Kernel.

1. $\Phi(x,t) > 0$ for all $x \in \mathbb{R}$, $t > 0$.
2. $\int_{-\infty}^{\infty} \Phi(x,t)\, dx = 1$ for all $t > 0$.
3. $\Phi(\cdot,t) \to \delta(x)$ as $t \to 0^+$.
4. The heat equation has infinite speed of propagation: if $f(x) \ge 0$ and $f \not\equiv 0$, then $u(x,t) > 0$ for all $x \in \mathbb{R}$, $t > 0$.

Semi-Infinite Domain. For $u_t = ku_{xx}$ on $x > 0$ with $u(0,t) = 0$:

$$u(x,t) = \int_0^{\infty} [\Phi(x-y,t) - \Phi(x+y,t)] f(y)\, dy$$

(odd extension / method of images).

15 Duhamel's Principle

Statement. The solution to

$$u_t = ku_{xx} + Q(x,t), \quad u(x,0) = 0, \quad \text{(with homogeneous BCs)}$$

can be written as

$$u(x,t) = \int_0^t w(x,t;\tau)\, d\tau,$$

where $w(x,t;\tau)$ solves the homogeneous heat equation for $t > \tau$ with initial condition $w(x,\tau;\tau) = Q(x,\tau)$ and the same homogeneous BCs.

Example. Solve $u_t = ku_{xx}$, $-\infty < x < \infty$, $u(x,0) = \delta(x)$ (point source).
Solution. Using the heat kernel:

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t)\delta(y)\, dy = \Phi(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}.$$

This is a Gaussian that spreads over time: width $\sim \sqrt{4kt}$, height $\sim 1/\sqrt{4\pi kt}$.

Example. Solve $u_t = ku_{xx}$, $-\infty < x < \infty$, $u(x,0) = H(x)$ (Heaviside).
Solution.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} e^{-(x-y)^2/(4kt)} dy.$$

Let $\eta = \frac{y-x}{\sqrt{4kt}}$:

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4kt}}^{\infty} e^{-\eta^2} d\eta = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right],$$

where $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\eta^2} d\eta$ is the error function.

THE WAVE EQUATION

16 Derivation

The one-dimensional wave equation models vibrations of a string:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0,$$

where $u(x,t)$ is the displacement and $c = \sqrt{T/\rho}$ is the wave speed (T = tension, ρ = linear mass density).

17 Separation of Variables

Problem. Solve

$$u_{tt} = c^2 u_{xx}, \quad u(0,t) = u(L,t) = 0,\ u(x,0) = f(x),\ u_t(x,0) = g(x).$$

Solution. Let $u(x,t) = X(x)T(t)$. Separating:

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda.$$

The eigenvalue problem is $X'' + \lambda X = 0$, $X(0) = X(L) = 0$, giving

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \sin \frac{n\pi x}{L}.$$

The time equation $T'' + c^2\lambda_n T = 0$ has the solution $T_n(t) = A_n \cos \frac{cn\pi t}{L} + B_n \sin \frac{cn\pi t}{L}$. By superposition:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{cn\pi t}{L} + B_n \sin \frac{cn\pi t}{L}\right) \sin \frac{n\pi x}{L},$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \quad B_n = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx.$$

Example. Solve $u_{tt} = 4u_{xx}$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = \sin 3x$, $u_t(x,0) = 0$.
Solution. Here $c = 2$, $L = \pi$. The IC is already $\sin 3x$, so $A_3 = 1$, all other $A_n = 0$, and all $B_n = 0$. The solution is

$$u(x,t) = \cos(6t)\sin(3x).$$

This is a standing wave with frequency $\omega_3 = 6$ and wavelength $2\pi/3$.

Example. A string of length $L = 1$ with $c = 1$ is plucked at the center:

$$u(x,0) = f(x) = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 2(1-x), & 1/2 \le x \le 1 \end{cases}, \quad u_t(x,0) = 0.$$

Solution. All $B_n = 0$. For A_n:

$$A_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx = \frac{8\sin(n\pi/2)}{n^2\pi^2} = \begin{cases} 0, & n \text{ even} \\ \frac{8(-1)^{(n-1)/2}}{n^2\pi^2}, & n \text{ odd} \end{cases}$$

Thus,

$$u(x,t) = \frac{8}{\pi^2}\left(\sin(\pi x)\cos(\pi t) - \frac{1}{9}\sin(3\pi x)\cos(3\pi t) + \cdots\right).$$

18 D'Alembert's Formula

Problem. Solve the wave equation on the whole line:

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty,\ t > 0,\ u(x,0) = f(x),\ u_t(x,0) = g(x).$$

Solution. The general solution of the wave equation is $u(x,t) = F(x+ct) + G(x-ct)$, where F and G are arbitrary C^2 functions. Applying the initial conditions yields D'Alembert's formula:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

Domain of Dependence. The value $u(x_0,t_0)$ depends only on values of f and g in the interval $[x_0 - ct_0,\ x_0 + ct_0]$.

Domain of Influence. The initial data at a point x_0 influences the solution in the cone $\{(x,t) : x_0 - ct \le x \le x_0 + ct\}$.

Finite Speed of Propagation. Disturbances travel at speed c, no faster. This is a fundamental difference between hyperbolic and parabolic equations.

Example. Solve $u_{tt} = c^2 u_{xx}$, $u(x,0) = e^{-x^2}$, $u_t(x,0) = 0$.
Solution. By D'Alembert's formula (with $g \equiv 0$),

$$u(x,t) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right].$$

The initial Gaussian splits into two half-amplitude pulses traveling in opposite directions at speed c.

Example. Solve $u_{tt} = u_{xx}$, $u(x,0) = 0$, $u_t(x,0) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| \ge 1 \end{cases}$.
Solution. Here $c = 1$, $f \equiv 0$. By D'Alembert:

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds = \frac{1}{2}(\text{length of overlap between } [x-t, x+t] \text{ and } [-1,1]).$$

For example, at $(x,t) = (0,3)$: the interval $[-3,3]$ contains $[-1,1]$ entirely, so $u(0,3) = \frac{1}{2}\cdot 2 = 1$.

19 Reflections (Semi-Infinite String)

Fixed End ($u(0,t) = 0$). Use the odd extension of f and g to $(-\infty,\infty)$ and apply D'Alembert's formula.

Free End ($u_x(0,t) = 0$). Use the even extension of f and g and apply D'Alembert's formula.

Example. A semi-infinite string $x > 0$ with $u(0,t) = 0$, $u(x,0) = e^{-(x-5)^2}$, $u_t(x,0) = 0$.
Solution. The odd extension of f is $f_{\text{odd}}(x) = e^{-(x-5)^2} - e^{-(x+5)^2}$. By D'Alembert:

$$u(x,t) = \frac{1}{2}\left[e^{-(x+ct-5)^2} - e^{-(x+ct+5)^2} + e^{-(x-ct-5)^2} - e^{-(x-ct+5)^2}\right]$$

for $x > 0$. The left-traveling pulse reflects off the fixed end at $x = 0$ with an inverted sign.

20 Energy and Uniqueness

Energy. The total energy of the vibrating string is

$$E(t) = \frac{1}{2}\int_0^L \left[u_t^2 + c^2 u_x^2\right]dx.$$

Theorem 2 (Conservation of Energy). For the wave equation with homogeneous Dirichlet or Neumann BCs, $E(t) = E(0)$ for all $t > 0$.

Proof. Differentiate $E(t)$:

$$\frac{dE}{dt} = \int_0^L \left[u_t u_{tt} + c^2 u_x u_{xt}\right]dx = \int_0^L u_t \underbrace{(u_{tt} - c^2 u_{xx})}_{0}\,dx + c^2[u_t u_x]_0^L = 0,$$

where the last term vanishes due to the homogeneous BCs.

Uniqueness. As a consequence, if u_1 and u_2 are two solutions with the same BCs, ICs, and forcing, then $w = u_1 - u_2$ has zero energy, hence $w_t = w_x = 0$, so $w \equiv 0$.

21 Non-Homogeneous Wave Equation

The problem $u_{tt} = c^2 u_{xx} + Q(x,t)$ can be solved by eigenfunction expansion (same as for the heat equation) or by Duhamel's principle.

Duhamel's Principle. The solution to

$$u_{tt} = c^2 u_{xx} + Q(x,t), \quad u(x,0) = 0,\ u_t(x,0) = 0$$

is $u(x,t) = \int_0^t w(x,t;\tau)\,d\tau$, where w solves the homogeneous wave equation for $t > \tau$ with $w(x,\tau;\tau) = 0$ and $w_t(x,\tau;\tau) = Q(x,\tau)$.

22 Wave Equation in Higher Dimensions

Two Dimensions.

$$u_{tt} = c^2(u_{xx} + u_{yy}) = c^2\nabla^2 u.$$

Kirchhoff's Formula (3D). For $u_{tt} = c^2\nabla^2 u$ in $\mathbb{R}^3$ with $u(\mathbf{x},0) = f(\mathbf{x})$, $u_t(\mathbf{x},0) = g(\mathbf{x})$:

$$u(\mathbf{x},t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\iint_{|\mathbf{y}-\mathbf{x}|=ct} f(\mathbf{y})\,dS\right] + \frac{1}{4\pi c^2 t}\iint_{|\mathbf{y}-\mathbf{x}|=ct} g(\mathbf{y})\,dS.$$

This exhibits **Huygens' principle**: signals propagate on a sharp wavefront (no residual effects).

Poisson's Formula (2D). Obtained from Kirchhoff's formula via the method of descent:

$$u(\mathbf{x},t) = \frac{1}{2\pi c}\frac{\partial}{\partial t}\iint_{|\mathbf{y}-\mathbf{x}|\le ct}\frac{f(\mathbf{y})}{\sqrt{c^2t^2 - |\mathbf{y}-\mathbf{x}|^2}}\,dA + \frac{1}{2\pi c}\iint_{|\mathbf{y}-\mathbf{x}|\le ct}\frac{g(\mathbf{y})}{\sqrt{c^2t^2 - |\mathbf{y}-\mathbf{x}|^2}}\,dA.$$

In 2D, signals leave residual effects (no Huygens' principle).

Damped Wave Equation. The equation $u_{tt} + 2\beta u_t = c^2 u_{xx}$ (with $\beta > 0$) models damped vibrations. Separation of variables gives time-dependent parts of the form $e^{-\beta t}(A\cos\omega_n t + B\sin\omega_n t)$, where $\omega_n = \sqrt{c^2\lambda_n - \beta^2}$ (assuming underdamping $c^2\lambda_n > \beta^2$). The energy $E(t)$ is no longer conserved but decreases monotonically.

LAPLACE'S AND POISSON'S EQUATIONS

23 Definitions

Laplace's Equation. $\nabla^2 u = 0$, where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ (2D) or $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (3D). Solutions are called **harmonic functions**.

Poisson's Equation. $\nabla^2 u = f(x,y)$, a non-homogeneous version of Laplace's equation.

Physical Interpretations. Steady-state heat distribution, electrostatic potential, gravitational potential, incompressible fluid flow.

24 Properties of Harmonic Functions

Theorem 3 (Mean Value Property). If u is harmonic in a domain Ω, then for any ball $B_r(\mathbf{x}_0) \subset \Omega$, the value at the center equals the average over the boundary:

$$u(\mathbf{x}_0) = \frac{1}{|\partial B_r|}\int_{\partial B_r(\mathbf{x}_0)} u\,dS.$$

In 2D: $u(x_0,y_0) = \frac{1}{2\pi r}\oint_{|(\xi,\eta)-(x_0,y_0)|=r} u(\xi,\eta)\,ds$.

Theorem 4 (Strong Maximum Principle). If u is harmonic in a connected, bounded domain Ω and continuous on $\overline{\Omega}$, then:

1. u attains its maximum (and minimum) on $\partial\Omega$.
2. If u attains its maximum at an interior point, then u is constant on Ω.

Consequences.

1. **Uniqueness** for the Dirichlet problem $\nabla^2 u = 0$ in Ω, $u = g$ on $\partial\Omega$.
2. **Continuous dependence** on boundary data.
3. A nonconstant harmonic function has **no local extrema** in the interior.

25 Laplace's Equation on a Rectangle

Problem. Solve $u_{xx} + u_{yy} = 0$ on $0 < x < a$, $0 < y < b$, with $u(x, 0) = 0$, $u(x, b) = 0$, $u(0, y) = 0$, $u(a, y) = f(y)$.

Solution. Separation of variables $u(x, y) = X(x)Y(y)$ gives

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

The Y problem: $Y'' + \lambda Y = 0$, $Y(0) = Y(b) = 0$, yields

$$\lambda_n = \frac{n^2\pi^2}{b^2}, \quad Y_n(y) = \sin\frac{n\pi y}{b}.$$

The X problem: $X'' - \lambda_n X = 0$, $X(0) = 0$, gives $X_n(x) = \sinh\frac{n\pi x}{b}$. The solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \frac{\sinh\frac{n\pi x}{b}}{\sinh\frac{n\pi a}{b}} \sin\frac{n\pi y}{b},$$

where $B_n = \frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy$.

Remark. If non-homogeneous BCs appear on multiple sides, decompose the problem into subproblems, each having non-homogeneous BCs on one side only, and add the solutions (superposition).

Example. Solve $u_{xx} + u_{yy} = 0$ on the unit square $[0, 1] \times [0, 1]$ with $u(x, 0) = 0$, $u(x, 1) = 0$, $u(0, y) = 0$, $u(1, y) = \sin(\pi y)$.
Solution. Here $a = b = 1$ and $f(y) = \sin(\pi y)$. Since $f(y)$ is already the first eigenfunction, $B_1 = 1$ and all other $B_n = 0$. Thus,

$$u(x, y) = \frac{\sinh(\pi x)}{\sinh(\pi)}\sin(\pi y).$$

Example (Multiple Non-Homogeneous Sides). Solve $\nabla^2 u = 0$ on $[0, 1]^2$ with $u(x, 0) = x$, $u(x, 1) = 0$, $u(0, y) = 0$, $u(1, y) = 1 - y$.
Solution. Decompose $u = u_1 + u_2$, where u_1 satisfies the problem with only the bottom BC $u_1(x, 0) = x$ non-homogeneous, and u_2 satisfies the problem with only the right BC $u_2(1, y) = 1 - y$ non-homogeneous. Solve each by the standard method and add.

26 Laplace's Equation on a Disk

Problem. Solve $\nabla^2 u = 0$ in the disk $r < R$, with $u(R, \theta) = f(\theta)$.

In polar coordinates, Laplace's equation is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Separation of variables $u(r, \theta) = \mathcal{R}(r)\Theta(\theta)$ yields

$$r^2\mathcal{R}'' + r\mathcal{R}' - n^2\mathcal{R} = 0 \quad \text{(Cauchy-Euler)}, \quad \Theta'' + n^2\Theta = 0.$$

Requiring periodicity $\Theta(\theta + 2\pi) = \Theta(\theta)$ gives $n = 0, 1, 2, \ldots$. The bounded solutions for $r < R$ are $\mathcal{R}_n(r) = r^n$. The solution is

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (a_n\cos n\theta + b_n\sin n\theta),$$

where a_n, b_n are the Fourier coefficients of $f(\theta)$.

Example. Solve $\nabla^2 u = 0$ in $r < 1$ with $u(1, \theta) = \cos^2\theta$.
Solution. $R = 1$. Rewrite: $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$. So $a_0/2 = 1/2$, $a_2 = 1/2$, all others zero. Thus,

$$u(r, \theta) = \frac{1}{2} + \frac{1}{2}r^2\cos 2\theta.$$

Verification: $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 2\cos 2\theta/2 + 2\cos 2\theta/(2) - 4r^2\cos 2\theta/(2r^2) = 0$. ✓

Example (Exterior Problem). Solve $\nabla^2 u = 0$ for $r > R$, $u(R, \theta) = f(\theta)$, $u \to 0$ as $r \to \infty$.
Solution. Boundedness as $r \to \infty$ requires $\mathcal{R}_n(r) = r^{-n}$ (instead of r^n):

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n (a_n\cos n\theta + b_n\sin n\theta).$$

27 Poisson's Integral Formula

For the disk $r < R$:

$$u(r, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}f(\phi)\,d\phi.$$

The kernel $P(r, \theta - \phi) = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta-\phi) + r^2}$ is called the **Poisson kernel**.

28 Laplace's Equation in 3D

Spherical Coordinates. In spherical coordinates (r, θ, ϕ), Laplace's equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0.$$

For axially symmetric problems (u independent of ϕ), separation of variables leads to Legendre's equation for the angular part:

$$u(r, \theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta),$$

where P_n are the Legendre polynomials.

For the interior of a sphere of radius R with $u(R, \theta) = f(\theta)$:

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n\left(\frac{r}{R}\right)^n P_n(\cos\theta), \quad A_n = \frac{2n+1}{2}\int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta\,d\theta.$$

29 Green's Functions for Laplace's Equation

Fundamental Solution. The fundamental solution of $\nabla^2 u = \delta(\mathbf{x} - \mathbf{x}_0)$ is

$$\Phi(\mathbf{x}, \mathbf{x}_0) = \begin{cases} -\frac{1}{2\pi}\ln|\mathbf{x} - \mathbf{x}_0|, & \text{in } \mathbb{R}^2 \\ \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|}, & \text{in } \mathbb{R}^3 \end{cases}$$

Green's Function. The Green's function $G(\mathbf{x}, \mathbf{x}_0)$ for $\nabla^2 u = 0$ in Ω with $u = 0$ on $\partial\Omega$ satisfies:

1. $\nabla^2 G = \delta(\mathbf{x} - \mathbf{x}_0)$ in Ω.
2. $G = 0$ on $\partial\Omega$.

The solution to $\nabla^2 u = f$ in Ω with $u = g$ on $\partial\Omega$ is

$$u(\mathbf{x}_0) = \int_{\Omega} G(\mathbf{x}, \mathbf{x}_0)f(\mathbf{x})\,d\mathbf{x} - \int_{\partial\Omega} g(\mathbf{x})\frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{x}_0)\,dS.$$

Method of Images. For the half-plane $y > 0$:

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{2\pi}\ln|\mathbf{x} - \mathbf{x}_0| + \frac{1}{2\pi}\ln|\mathbf{x} - \mathbf{x}_0^*|,$$

where $\mathbf{x}_0^*$ is the reflection of $\mathbf{x}_0$ across the boundary (for $\mathbf{x}_0 = (x_0, y_0)$, $\mathbf{x}_0^* = (x_0, -y_0)$).

Green's Identities. Let u, v be C^2 functions on $\overline{\Omega}$:

1. First: $\int_{\Omega}(v\nabla^2 u + \nabla v \cdot \nabla u)\,dV = \int_{\partial\Omega} v\frac{\partial u}{\partial n}\,dS$.
2. Second: $\int_{\Omega}(v\nabla^2 u - u\nabla^2 v)\,dV = \int_{\partial\Omega}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)dS$.

Green's Representation Formula. If u is harmonic in Ω:

$$u(\mathbf{x}_0) = \int_{\partial\Omega}\left[\Phi(\mathbf{x}, \mathbf{x}_0)\frac{\partial u}{\partial n}(\mathbf{x}) - u(\mathbf{x})\frac{\partial\Phi}{\partial n}(\mathbf{x}, \mathbf{x}_0)\right]dS(\mathbf{x}).$$

Example. Find the Green's function for Laplace's equation in the upper half-plane $y > 0$ with $u(x, 0) = f(x)$.
Solution. Place the image source at $(x_0, -y_0)$:

$$\begin{aligned} G((x, y), (x_0, y_0)) = &-\frac{1}{2\pi}\ln\sqrt{(x - x_0)^2 + (y - y_0)^2} \\ &+ \frac{1}{2\pi}\ln\sqrt{(x - x_0)^2 + (y + y_0)^2}. \end{aligned}$$

The solution is

$$u(x_0, y_0) = -\int_{-\infty}^{\infty} f(x)\frac{\partial G}{\partial y}\Big|_{y=0}dx = \frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{(x - x_0)^2 + y_0^2}\,dx,$$

which is the Poisson integral formula for the half-plane.

FOURIER TRANSFORM METHODS

30 The Fourier Transform

Definition. The Fourier transform of $f(x)$ is

$$\hat{f}(\xi) = \mathcal{F}\{f\} = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx.$$

The inverse Fourier transform is

$$f(x) = \mathcal{F}^{-1}\left\{\hat{f}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}\,d\xi.$$

Properties.

1. Linearity: $\mathcal{F}\{\alpha f + \beta g\} = \alpha\hat{f} + \beta\hat{g}$.
2. Derivative: $\mathcal{F}\{f'\} = i\xi\hat{f}(\xi)$.

3. Higher derivatives: $\mathcal{F}\{f^{(n)}\} = (i\xi)^n \hat{f}(\xi)$.
4. Convolution: $\mathcal{F}\{f * g\} = \hat{f} \cdot \hat{g}$.
5. Shift: $\mathcal{F}\{f(x-a)\} = e^{-ia\xi}\hat{f}(\xi)$.
6. Modulation: $\mathcal{F}\{e^{iax}f(x)\} = \hat{f}(\xi - a)$.
7. Scaling: $\mathcal{F}\{f(ax)\} = \frac{1}{|a|}\hat{f}(\xi/a)$.

Parseval's Theorem.

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\,d\xi.$$

31 Solving PDEs via the Fourier Transform

Heat Equation on $\mathbb{R}$. For $u_t = ku_{xx}$, $u(x,0) = f(x)$, apply $\mathcal{F}\{\cdot\}$ in x:

$$\hat{u}_t = -k\xi^2\hat{u} \quad \Rightarrow \quad \hat{u}(\xi,t) = \hat{f}(\xi)e^{-k\xi^2 t}.$$

Inverting: $u(x,t) = f * \Phi$, where $\Phi(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)}$ is the heat kernel.

Wave Equation on $\mathbb{R}$. For $u_{tt} = c^2u_{xx}$, $u(x,0) = f(x)$, $u_t(x,0) = g(x)$:

$$\hat{u}_{tt} = -c^2\xi^2\hat{u} \quad \Rightarrow \quad \hat{u} = \hat{f}\cos(c\xi t) + \frac{\hat{g}}{c\xi}\sin(c\xi t).$$

Inverting recovers D'Alembert's formula.

Example. Solve $u_t = ku_{xx}$, $-\infty < x < \infty$, $u(x,0) = e^{-|x|}$.
Solution. First, $\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-|x|}e^{-i\xi x}dx = \frac{2}{1+\xi^2}$. Then,

$$\hat{u}(\xi,t) = \frac{2}{1+\xi^2}e^{-k\xi^2 t}.$$

Inverting:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1+\xi^2}e^{-k\xi^2 t}e^{i\xi x}\,d\xi.$$

For small t, this is close to $e^{-|x|}$. For large t, the Gaussian factor dominates and $u \approx \frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)} \cdot 2$.

Example. Solve $u_t = ku_{xx}$, $x > 0$, $u(0,t) = 0$, $u(x,0) = e^{-x}$.
Solution. Use the Fourier sine transform. Since $u(0,t) = 0$ (Dirichlet), we have

$$\hat{u}_{s,t} = -k\xi^2\hat{u}_s + k\xi\underbrace{u(0,t)}_{=0} = -k\xi^2\hat{u}_s.$$

So $\hat{u}_s(\xi,t) = \hat{f}_s(\xi)e^{-k\xi^2 t}$, where $\hat{f}_s(\xi) = \int_0^{\infty} e^{-x}\sin(\xi x)\,dx = \frac{\xi}{1+\xi^2}$.

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty} \frac{\xi}{1+\xi^2}e^{-k\xi^2 t}\sin(\xi x)\,d\xi.$$

32 Fourier Sine and Cosine Transforms

For problems on the half-line $x > 0$:

Fourier Sine Transform.

$$\hat{f}_s(\xi) = \int_0^{\infty} f(x)\sin(\xi x)\,dx, \quad f(x) = \frac{2}{\pi}\int_0^{\infty} \hat{f}_s(\xi)\sin(\xi x)\,d\xi.$$

Useful for Dirichlet BCs at $x = 0$: $\mathcal{F}\{f''\}_s = -\xi^2\hat{f}_s + \xi f(0)$.

Fourier Cosine Transform.

$$\hat{f}_c(\xi) = \int_0^{\infty} f(x)\cos(\xi x)\,dx, \quad f(x) = \frac{2}{\pi}\int_0^{\infty} \hat{f}_c(\xi)\cos(\xi x)\,d\xi.$$

Useful for Neumann BCs at $x = 0$: $\mathcal{F}\{f''\}_c = -\xi^2\hat{f}_c - f'(0)$.

Common Fourier Transform Pairs.

$f(x)$	$\hat{f}(\xi)$
$e^{-a\|x\|}$ $(a > 0)$	$\frac{2a}{a^2+\xi^2}$
e^{-ax^2}	$\sqrt{\frac{\pi}{a}}e^{-\xi^2/(4a)}$
$\delta(x-a)$	$e^{-ia\xi}$
$\text{rect}(x)$	$\frac{2\sin(\xi/2)}{\xi}$
$\text{sgn}(x)$	$\frac{2}{i\xi}$

STURM-LIOUVILLE THEORY AND EIGENFUNCTION EXPANSIONS

33 Sturm-Liouville Problem (Review)

Standard Form. A Sturm-Liouville problem is a BVP of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x)y = 0, \quad a < x < b,$$

with boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0,$$

where $p(x) > 0$, $r(x) > 0$, $q(x) \geq 0$ on $[a,b]$.

Key Properties.

1. All eigenvalues λ_n are real, $\lambda_1 < \lambda_2 < \cdots$, $\lambda_n \to \infty$.
2. Eigenfunctions ϕ_n are orthogonal w.r.t. weight $r(x)$: $\int_a^b r(x)\phi_m(x)\phi_n(x)\,dx = 0$ for $m \neq n$.
3. Completeness: any piecewise smooth f can be expanded as

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \quad c_n = \frac{\int_a^b r(x)f(x)\phi_n(x)\,dx}{\int_a^b r(x)[\phi_n(x)]^2\,dx}.$$

34 Common Eigenvalue Problems

The following table summarizes common eigenvalue problems arising from separation of variables.

BCs	**Eigenvalues**	**Eigenfunctions**
$X(0) = X(L) = 0$	$\left(\frac{n\pi}{L}\right)^2$	$\sin\frac{n\pi x}{L}$
$X'(0) = X'(L) = 0$	$\left(\frac{n\pi}{L}\right)^2$	$\cos\frac{n\pi x}{L}$
$X(0) = 0$ $X'(L) = 0$	$\left(\frac{(2n-1)\pi}{2L}\right)^2$	$\sin\frac{(2n-1)\pi x}{2L}$
$X(-L) = X(L)$ $X'(-L) = X'(L)$	$\left(\frac{n\pi}{L}\right)^2$	$\cos\frac{n\pi x}{L}$, $\sin\frac{n\pi x}{L}$

35 Singular Sturm-Liouville Problems

When $p(a) = 0$ or $p(b) = 0$, or the interval is unbounded, the problem is singular. A boundary condition is replaced by a **boundedness condition**. Examples:

- **Bessel's equation** on $[0,R]$: $p(x) = x$ vanishes at $x = 0$. Replace the BC at $x = 0$ by requiring $|y(0)| < \infty$.
- **Legendre's equation** on $[-1,1]$: $p(x) = 1 - x^2$ vanishes at $x = \pm 1$. Require $|y(\pm 1)| < \infty$.

Rayleigh Quotient. For the Sturm-Liouville problem, the eigenvalues satisfy

$$\lambda = \frac{-p(x)y\,y'\Big|_a^b + \int_a^b \left[p(x)(y')^2 + q(x)y^2\right]dx}{\int_a^b r(x)y^2\,dx}.$$

This shows $\lambda \geq 0$ when $q \geq 0$ and the boundary terms are non-negative.

Example. Find the eigenvalues and eigenfunctions of $y'' + \lambda y = 0$, $y(0) = 0$, $y'(L) = 0$.
Solution. For $\lambda > 0$: $y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$. $y(0) = 0 \Rightarrow A = 0$. $y'(L) = B\sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0$. For nontrivial B:

$$\sqrt{\lambda}L = \frac{(2n-1)\pi}{2} \quad \Rightarrow \quad \lambda_n = \frac{(2n-1)^2\pi^2}{4L^2},$$

where

$$\phi_n(x) = \sin\frac{(2n-1)\pi x}{2L}, \quad n = 1, 2, \ldots$$

Example. Expand $f(x) = 1$ on $[0,L]$ in the eigenfunctions $\phi_n(x) = \sin\frac{(2n-1)\pi x}{2L}$.
Solution.

$$c_n = \frac{\int_0^L \sin\frac{(2n-1)\pi x}{2L}\,dx}{\int_0^L \sin^2\frac{(2n-1)\pi x}{2L}\,dx} = \frac{\frac{2L}{(2n-1)\pi}}{L/2} = \frac{4}{(2n-1)\pi}$$

Thus, $1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{2L}$ on $[0,L]$.

HIGHER-DIMENSIONAL PROBLEMS

36 Heat Equation in Higher Dimensions

Two Dimensions (Rectangle). Solve $u_t = k(u_{xx} + u_{yy})$ on $[0,a] \times [0,b]$ with Dirichlet BCs. Let $u(x,y,t) = X(x)Y(y)T(t)$. Double separation of variables gives

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,\exp\left[-k\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)t\right],$$

where $B_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy$.

Example. Solve $u_t = u_{xx} + u_{yy}$ on $[0,1]\times[0,1]$, $u = 0$ on boundary, $u(x,y,0) = \sin(\pi x)\sin(2\pi y)$.
Solution. $a = b = 1$, $k = 1$. The IC matches $(m,n) = (1,2)$, so $B_{12} = 1$, all others zero. Thus,

$$u(x,y,t) = \sin(\pi x)\sin(2\pi y)\,e^{-(1+4)\pi^2 t} = \sin(\pi x)\sin(2\pi y)\,e^{-5\pi^2 t}.$$

37 Heat Equation on a Disk

Problem. $u_t = k\nabla^2 u$ in $r < R$ with $u(R,\theta,t) = 0$, $u(r,\theta,0) = f(r,\theta)$.

In polar coordinates: $u_t = k\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right)$.

Separation $u = \mathcal{R}(r)\Theta(\theta)T(t)$ leads to:

- $\Theta'' + n^2\Theta = 0$: $\Theta_n = \{A_n\cos n\theta, B_n\sin n\theta\}$, $n = 0,1,2,\ldots$
- $r^2\mathcal{R}'' + r\mathcal{R}' + (r^2\lambda - n^2)\mathcal{R} = 0$: Bessel's equation, giving $\mathcal{R}(r) = J_n(\sqrt{\lambda}\,r)$, bounded at $r = 0$.
- BC $\mathcal{R}(R) = 0$: $J_n(\sqrt{\lambda}\,R) = 0$, so $\sqrt{\lambda_{nm}}R = j_{nm}$ (the m-th zero of J_n).

The solution is

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{j_{nm}r}{R}\right)(A_{nm}\cos n\theta + B_{nm}\sin n\theta)\,e^{-k(j_{nm}/R)^2 t}.$$

38 Wave Equation on a Rectangular Membrane

Solve $u_{tt} = c^2(u_{xx} + u_{yy})$ on $[0,a]\times[0,b]$ with $u = 0$ on the boundary. The natural frequencies are

$$\omega_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \quad m,n = 1,2,3,\ldots$$

and the mode shapes are $\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$.

39 Wave Equation on a Circular Membrane

For $u_{tt} = c^2\nabla^2 u$ in $r < R$ with $u(R,\theta,t) = 0$, separation of variables leads to Bessel functions in r. The natural frequencies are

$$\omega_{nm} = \frac{c j_{nm}}{R},$$

where j_{nm} is the m-th positive zero of J_n.

40 Laplace's Equation in a Cylinder

For $\nabla^2 u = 0$ in a cylinder $r < R$, $0 < z < H$, separation in cylindrical coordinates (r,θ,z) leads to the Bessel equation in r, trigonometric functions in θ, and hyperbolic functions in z:

$$u(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\lambda_{nm}r)(A_{nm}\cos n\theta + B_{nm}\sin n\theta)\sinh(\lambda_{nm}z).$$

Example. For a circular drum $r < 1$, $c = 1$, with $u(1,\theta,t) = 0$ and radially symmetric IC $u(r,0) = 1 - r^2$, $u_t(r,0) = 0$:
Solution. Radially symmetric means $n = 0$ (no θ dependence). The eigenfunctions are $J_0(j_{0m}r)$, where j_{0m} are the zeros of J_0 ($j_{01}\approx 2.405$, $j_{02}\approx 5.520$, $\ldots$). The solution is

$$u(r,t) = \sum_{m=1}^{\infty} A_m J_0(j_{0m}r)\cos(j_{0m}t),$$

where $A_m = \frac{2}{[J_1(j_{0m})]^2}\int_0^1 r(1 - r^2)J_0(j_{0m}r)\,dr$.

SPECIAL TOPICS

41 Separation of Variables: General Strategy

1. Write u as a product of functions of single variables.
2. Substitute into the PDE and separate into ODEs connected by separation constants.
3. Apply homogeneous BCs to obtain an eigenvalue problem.
4. Solve the eigenvalue problem for eigenvalues λ_n and eigenfunctions.
5. Solve the remaining ODE(s) using the eigenvalues λ_n.
6. Form the general solution by superposition.
7. Apply initial/remaining conditions to determine coefficients (via Fourier series).

42 Characteristics and the General Wave Equation

For the general second-order PDE $Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0$ with $\Delta = B^2 - AC > 0$ (hyperbolic), the change of variables

$$\xi = \phi(x,y), \quad \eta = \psi(x,y),$$

where ϕ and ψ are constant along the two families of characteristics, transforms the PDE to

$$u_{\xi\eta} = \text{(lower order terms)}.$$

43 Uniqueness Theorems Summary

Equation	Uniqueness Method
Heat equation	Maximum principle
Wave equation	Energy methods
Laplace's equation	Maximum principle

44 Comparison of the Three Fundamental PDEs

	Heat	Wave	Laplace
Type	Parabolic	Hyperbolic	Elliptic
Propagation	Infinite speed	Finite speed (c)	Steady-state
Smoothing	Yes	No	Yes (analytic)
Reversible	No	Yes	N/A
Max. princ.	Yes	No	Yes
Energy	Dissipates	Conserved	N/A

45 Change of Variables to Reduce PDEs

Example. Reduce $u_{tt} + 3u_t = c^2u_{xx}$ (damped wave) to a standard form.
Solution. Let $u(x,t) = e^{-3t/2}v(x,t)$. Then $u_t = e^{-3t/2}(v_t - \frac{3}{2}v)$, $u_{tt} = e^{-3t/2}(v_{tt} - 3v_t + \frac{9}{4}v)$. Substituting:

$$v_{tt} - 3v_t + \tfrac{9}{4}v + 3v_t - \tfrac{9}{2}v = c^2v_{xx} \quad\Rightarrow\quad v_{tt} = c^2v_{xx} + \tfrac{9}{4}v.$$

This is a Klein-Gordon equation, solvable by separation of variables.

Example. Reduce $u_t + xu_x = 0$ using characteristics.
Solution. Characteristics: $dx/dt = x$, so $x = x_0e^t$, i.e., $x_0 = xe^{-t}$. The solution is $u(x,t) = f(xe^{-t})$ for any f, with IC determining f.

Similarity Solutions. For the heat equation $u_t = ku_{xx}$, the combination $\eta = x/\sqrt{4kt}$ is a similarity variable. If $u(x,t) = v(\eta)$, then v satisfies the ODE

$$v'' + 2\eta v' = 0 \quad\Rightarrow\quad v(\eta) = A + B\,\mathrm{erf}(\eta).$$

This yields the error-function solutions used for semi-infinite domain problems.

46 Summary of Solution Methods

Method	When to Use
Separation of variables	Bounded domains, homog. BCs
Eigenfunction expansion	Non-homog. source terms
Fourier transform	Infinite domains ($\mathbb{R}$)
Fourier sine/cosine	Semi-infinite ($x > 0$)
D'Alembert's formula	Wave equation on $\mathbb{R}$
Green's functions	General BVPs
Duhamel's principle	Non-homog. with zero IC
Method of images	Simple domain symmetries
Method of characteristics	First-order PDEs
Change of variables	Reducing to known forms
Laplace/Fourier in t	IVPs with const. coefficients

NUMERICAL METHODS FOR PDEs

47 Finite Difference Basics

Grid Setup. Discretize the domain with spatial step $\Delta x = h$ and time step $\Delta t = k$. Grid points: $x_j = j\,h$, $t_n = n\,k$. Denote $U_j^n \approx u(x_j,t_n)$.

Difference Approximations.

$$u_x \approx \frac{U_{j+1}^n - U_{j-1}^n}{2h} \quad \text{(centered, } O(h^2))$$

$$u_x \approx \frac{U_{j+1}^n - U_j^n}{h} \quad \text{(forward, } O(h))$$

$$u_x \approx \frac{U_j^n - U_{j-1}^n}{h} \quad \text{(backward, } O(h))$$

$$u_{xx} \approx \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} \quad \text{(centered, } O(h^2))$$

$$u_t \approx \frac{U_j^{n+1} - U_j^n}{k} \quad \text{(forward, } O(k))$$

Higher-Order Approximations.

$$u_x \approx \frac{-U_{j+2}^n + 8U_{j+1}^n - 8U_{j-1}^n + U_{j-2}^n}{12h} \quad (O(h^4))$$

$$u_{xx} \approx \frac{-U_{j+2}^n + 16U_{j+1}^n - 30U_j^n + 16U_{j-1}^n - U_{j-2}^n}{12h^2} \quad (O(h^4))$$

48 Heat Equation: Explicit Method (FTCS)

Forward Time, Centered Space for $u_t = ku_{xx}$:

$$\frac{U_j^{n+1} - U_j^n}{k} = k_{\text{diff}} \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}$$

Let $r = k_{\text{diff}}\, k/h^2$ (the **mesh ratio** or **Courant number** for diffusion). Then:

$$\boxed{U_j^{n+1} = rU_{j-1}^n + (1-2r)U_j^n + rU_{j+1}^n}$$

Stability (von Neumann analysis). Substitute $U_j^n = g^n e^{i\xi jh}$. The amplification factor is

$$g(\xi) = 1 - 2r(1-\cos\xi h) = 1 - 4r\sin^2\frac{\xi h}{2}.$$

Stability requires $|g| \le 1$ for all ξ, which gives $\boxed{r \le \frac{1}{2}}$, i.e., $k \le \frac{h^2}{2k_{\text{diff}}}$.

Example. Solve $u_t = u_{xx}$ on $[0,1]$, $u(0,t) = u(1,t) = 0$, $u(x,0) = \sin(\pi x)$, using $h = 0.25$, $k = 0.02$ (so $r = 0.02/0.0625 = 0.32 \le 0.5$, stable).
Solution. Interior points: $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$. Initial values: $U_1^0 = \sin(\pi/4) = \frac{\sqrt{2}}{2}$, $U_2^0 = 1$, $U_3^0 = \frac{\sqrt{2}}{2}$. At $n = 1$:

$$U_1^1 = 0.32 \cdot 0 + (1-0.64)\cdot\tfrac{\sqrt{2}}{2} + 0.32\cdot 1 = 0.574$$
$$U_2^1 = 0.32\cdot\tfrac{\sqrt{2}}{2} + 0.36\cdot 1 + 0.32\cdot\tfrac{\sqrt{2}}{2} = 0.813$$
$$U_3^1 = U_1^1 = 0.574 \quad \text{(by symmetry)}$$

The exact solution is $u(x,t) = e^{-\pi^2 t}\sin(\pi x)$. At $t = 0.02$: $e^{-0.02\pi^2} \approx 0.821$, giving exact values 0.581, 0.821, 0.581. The numerical error is $\sim 1\%$.

49 Heat Equation: Implicit Method (BTCS)

Backward Time, Centered Space:

$$\frac{U_j^{n+1} - U_j^n}{k} = k_{\text{diff}} \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{h^2}$$

$$\boxed{-rU_{j-1}^{n+1} + (1+2r)U_j^{n+1} - rU_{j+1}^{n+1} = U_j^n}$$

This is a tridiagonal system $A\mathbf{U}^{n+1} = \mathbf{U}^n$ with

$$A = \begin{pmatrix} 1+2r & -r & & \\ -r & 1+2r & -r & \\ & \ddots & \ddots & \ddots \\ & & -r & 1+2r \end{pmatrix}.$$

Unconditionally stable ($|g| \le 1$ for all $r > 0$), first-order in time, second-order in space: $O(k + h^2)$.

Solving Tridiagonal Systems. The Thomas algorithm (LU factorization for tridiagonal matrices) solves the system in $O(N)$ operations:

1. Forward sweep: $c_i' = c_i/(b_i - a_i c_{i-1}')$, $d_i' = (d_i - a_i d_{i-1}')/(b_i - a_i c_{i-1}')$.
2. Back substitution: $x_N = d_N'$, $x_i = d_i' - c_i' x_{i+1}$.

Here $a_i = -r$ (lower), $b_i = 1+2r$ (diagonal), $c_i = -r$ (upper), $d_i = U_i^n$ (RHS).

50 Crank-Nicolson Method

Average of explicit and implicit (trapezoidal rule in time):

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{k_{\text{diff}}}{2}\left[\frac{\delta_x^2 U_j^{n+1}}{h^2} + \frac{\delta_x^2 U_j^n}{h^2}\right]$$

where $\delta_x^2 U_j = U_{j+1} - 2U_j + U_{j-1}$. This gives:

$$\boxed{-\frac{r}{2}U_{j-1}^{n+1} + (1+r)U_j^{n+1} - \frac{r}{2}U_{j+1}^{n+1} = \frac{r}{2}U_{j-1}^n + (1-r)U_j^n + \frac{r}{2}U_{j+1}^n}$$

Unconditionally stable, second-order in both time and space: $O(k^2 + h^2)$. This is the standard method for parabolic equations.

Example. Compare methods for $u_t = u_{xx}$, $u(x,0) = \sin(\pi x)$, $h = 0.1$, $k = 0.01$.
At $t = 0.01$ and $x = 0.5$: exact $= e^{-0.01\pi^2} \approx 0.9061$.

- FTCS ($r = 1.0$): **unstable** (blows up, since $r > 0.5$).
- BTCS ($r = 1.0$): $U \approx 0.9032$ (stable but $O(k)$ error).
- Crank-Nicolson ($r = 1.0$): $U \approx 0.9060$ (stable, $O(k^2)$ error).

51 Wave Equation: Explicit Method

For $u_{tt} = c^2 u_{xx}$, centered differences in both x and t:

$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{k^2} = c^2\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}$$

Let $\nu = ck/h$ (the **Courant number** or **CFL number**). Then:

$$\boxed{U_j^{n+1} = \nu^2 U_{j+1}^n + 2(1-\nu^2)U_j^n + \nu^2 U_{j-1}^n - U_j^{n-1}}$$

Stability: $\boxed{\nu \le 1}$ (the **CFL condition**: $k \le h/c$).

Remark. When $\nu = 1$ exactly, the numerical scheme reproduces the exact solution. This remarkable property follows from D'Alembert's formula.

Starting the Scheme. The method requires U_j^0 and U_j^1. From the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$:

$$U_j^0 = f(x_j), \quad U_j^1 \approx f(x_j) + kg(x_j) + \frac{k^2c^2}{2}f''(x_j)$$

using a Taylor expansion (second-order accurate start).

Example. Solve $u_{tt} = u_{xx}$ on $[0,1]$, $u(0,t) = u(1,t) = 0$, $u(x,0) = \sin(\pi x)$, $u_t(x,0) = 0$, with $h = 0.25$, $k = 0.25$ ($\nu = 1$).
Solution. Since $\nu = 1$, the scheme is exact. $U_j^0 : (0, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0)$. For U_j^1, using $g = 0$ and $f'' = -\pi^2\sin(\pi x)$:

$$U_j^1 = f(x_j) + \frac{k^2}{2}f''(x_j) = \sin(\pi x_j)\left(1 - \frac{\pi^2}{32}\right) \approx (0,\ 0.487,\ 0.692,\ 0.487,\ 0)$$

Compare exact: $u(x_j, 0.25) = \cos(0.25\pi)\sin(\pi x_j) \approx 0.707\sin(\pi x_j)$, giving $(0,\ 0.500,\ 0.707,\ 0.500,\ 0)$. The small discrepancy is from the coarse grid ($h = 0.25$).

52 Laplace's Equation: Iterative Methods

For $u_{xx} + u_{yy} = 0$ on a grid with $h = \Delta x = \Delta y$, the 5-point stencil is:

$$\boxed{U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j} = 0}$$

i.e., the value at each interior point is the average of its four neighbors.

Jacobi Iteration.

$$U_{i,j}^{(k+1)} = \frac{1}{4}\left(U_{i+1,j}^{(k)} + U_{i-1,j}^{(k)} + U_{i,j+1}^{(k)} + U_{i,j-1}^{(k)}\right)$$

Convergence rate: spectral radius $\rho \approx 1 - \pi^2h^2/2$ (slow for small h). Iterations needed: $O(h^{-2})$.

Gauss-Seidel Iteration. Use updated values as soon as available:

$$U_{i,j}^{(k+1)} = \frac{1}{4}\left(U_{i+1,j}^{(k)} + U_{i-1,j}^{(k+1)} + U_{i,j+1}^{(k)} + U_{i,j-1}^{(k+1)}\right)$$

Converges twice as fast as Jacobi: $\rho \approx 1 - \pi^2h^2$ (but still $O(h^{-2})$ iterations).

Successive Over-Relaxation (SOR). Accelerate Gauss-Seidel with a relaxation parameter ω:

$$U_{i,j}^{(k+1)} = (1-\omega)U_{i,j}^{(k)} + \frac{\omega}{4}\left(U_{i+1,j}^{(k)} + U_{i-1,j}^{(k+1)} + U_{i,j+1}^{(k)} + U_{i,j-1}^{(k+1)}\right)$$

Optimal parameter for a square domain: $\boxed{\omega_{\text{opt}} = \frac{2}{1+\sin(\pi h)} \approx 2 - 2\pi h}$.

With optimal ω, iterations $= O(h^{-1})$.

Example. Solve $\nabla^2 u = 0$ on $[0,1]^2$, $h = 1/3$ (4 interior points). BCs: $u = 0$ on left, bottom, top; $u = 1$ on right.
Solution. Interior points: $(1/3, 1/3)$, $(2/3, 1/3)$, $(1/3, 2/3)$, $(2/3, 2/3)$. By symmetry, $U_{1,1} = U_{1,2} = a$ and $U_{2,1} = U_{2,2} = b$. The 5-point equations give:

$$4a = 0 + b + 0 + 0 = b$$
$$4b = a + 1 + 0 + 0 = a + 1$$

Solving: $b = 4a$, $16a = a + 1 \Rightarrow a = 1/15$, $b = 4/15$. After 1 Jacobi iteration from $U = 0$: $a^{(1)} = 0$, $b^{(1)} = 1/4$. After 2: $a^{(2)} = 1/16$, $b^{(2)} = 1/4$. Converges to $a = 1/15 \approx 0.067$, $b = 4/15 \approx 0.267$.

53 Poisson's Equation

For $u_{xx} + u_{yy} = f(x,y)$, the 5-point stencil becomes:

$$U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j} = h^2 f_{i,j}$$

This gives a linear system $A\mathbf{U} = \mathbf{b}$. For an $N \times N$ interior grid, A is $N^2 \times N^2$, sparse, symmetric positive definite. Direct solvers (e.g., sparse LU) cost $O(N^3)$; iterative methods (SOR, conjugate gradient) are preferred for large N.

9-Point Stencil (higher accuracy, $O(h^4)$ for Laplace):

$$20U_{i,j} - 4(U_{i\pm1,j} + U_{i,j\pm1}) - (U_{i\pm1,j\pm1}) = 0$$

54 Consistency, Stability, and Convergence

Consistency. A finite difference scheme is consistent if the truncation error $\tau \to 0$ as $h, k \to 0$. Found by Taylor-expanding the scheme and comparing to the PDE.

Stability. The scheme does not amplify errors. Checked by **von Neumann analysis**: substitute $U_j^n = g^n e^{i\xi jh}$ into the scheme and require $|g(\xi)| \le 1$ for all ξ.

Lax Equivalence Theorem. For a consistent linear scheme,

$$\boxed{\text{Stability} \iff \text{Convergence}}$$

Convergence means $\max_j |U_j^n - u(x_j, t_n)| \to 0$ as $h, k \to 0$.

Lax-Wendroff Theorem. If a conservative scheme converges, the limit is a weak solution of the conservation law.

Truncation Error Summary.

Scheme	Order	Stability
FTCS (heat)	$O(k + h^2)$	$r \le 1/2$
BTCS (heat)	$O(k + h^2)$	Unconditional
Crank-Nicolson	$O(k^2 + h^2)$	Unconditional
Explicit (wave)	$O(k^2 + h^2)$	$\nu \le 1$
5-pt Laplace	$O(h^2)$	Always
9-pt Laplace	$O(h^4)$	Always

55 Advection Equation Schemes

For $u_t + au_x = 0$ $(a > 0)$, several schemes exist.

FTCS (centered space): $U_j^{n+1} = U_j^n - \frac{\nu}{2}(U_{j+1}^n - U_{j-1}^n)$, where $\nu = ak/h$. **Unconditionally unstable** (do not use).

Upwind (FTBS): Uses the direction of propagation:

$$\boxed{U_j^{n+1} = U_j^n - \nu(U_j^n - U_{j-1}^n) = (1-\nu)U_j^n + \nu U_{j-1}^n}$$

Stable for $0 \le \nu \le 1$. First-order: $O(k + h)$. Introduces numerical diffusion $\sim \frac{ah}{2}(1-\nu)u_{xx}$.

Lax-Friedrichs:

$$U_j^{n+1} = \frac{1}{2}(U_{j+1}^n + U_{j-1}^n) - \frac{\nu}{2}(U_{j+1}^n - U_{j-1}^n)$$

Stable for $\nu \le 1$. First-order, very diffusive.

Lax-Wendroff:

$$\boxed{U_j^{n+1} = U_j^n - \frac{\nu}{2}(U_{j+1}^n - U_{j-1}^n) + \frac{\nu^2}{2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n)}$$

Stable for $\nu \le 1$. Second-order: $O(k^2 + h^2)$. Less diffusion but can produce oscillations near discontinuities (**dispersive errors**).

Example. Solve $u_t + u_x = 0$, $u(x, 0) = e^{-100(x-0.5)^2}$, $h = 0.01$, $k = 0.005$ $(\nu = 0.5)$.
Solution. After 100 time steps $(t = 0.5)$, the exact solution is $u(x, 0.5) = e^{-100(x-1)^2}$. Numerical results:

- Upwind: peak ≈ 0.88 (smeared by numerical diffusion).
- Lax-Wendroff: peak ≈ 0.99 with small trailing oscillations.
- Exact: peak $= 1.0$.

56 Nonlinear Conservation Laws

For $u_t + f(u)_x = 0$, conservative schemes have the form

$$U_j^{n+1} = U_j^n - \frac{k}{h}(F_{j+1/2}^n - F_{j-1/2}^n)$$

where $F_{j+1/2}$ is a **numerical flux** consistent with f (i.e., $F(u, u) = f(u)$).

Lax-Friedrichs Flux: $F_{j+1/2} = \frac{1}{2}[f(U_j) + f(U_{j+1})] - \frac{h}{2k}(U_{j+1} - U_j)$.

Godunov Flux: $F_{j+1/2} = f(u^*)$, where u^* is the solution of the Riemann problem at $x_{j+1/2}$. Exact but expensive for systems.

Roe Flux: $F_{j+1/2} = \frac{1}{2}[f(U_j) + f(U_{j+1})] - \frac{1}{2}|\hat{a}|(U_{j+1} - U_j)$, where $\hat{a} = \frac{f(U_{j+1}) - f(U_j)}{U_{j+1} - U_j}$ (Roe-averaged speed).

Example. For Burgers' equation $u_t + (u^2/2)_x = 0$ with a shock, the Godunov scheme correctly captures the shock location and speed (satisfying Rankine-Hugoniot), while non-conservative schemes can produce incorrect shock speeds.

57 Method of Lines

Discretize in space only, leaving a system of ODEs in time:

$$\frac{dU_j}{dt} = k_{\text{diff}} \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} \quad \text{(heat equation)}$$

This system $\mathbf{U}'(t) = A\mathbf{U}(t)$ can be solved by any ODE integrator (Euler, RK4, etc.).

Advantages: Decouples spatial and temporal discretization. Can use adaptive time stepping and high-order time integrators.

RK4 + centered differences for $u_t = ku_{xx}$: 4th-order in time, 2nd-order in space, stability requires $r \le 0.69$ approximately.

Example. For the heat equation with N interior points, the method of lines gives $\mathbf{U}' = \frac{k_{\text{diff}}}{h^2} T\mathbf{U}$, where T is the tridiagonal matrix with -2 on the diagonal and 1 on the off-diagonals. The eigenvalues of T/h^2 are $-4\sin^2(m\pi h/2)/h^2$ for $m = 1, \ldots, N$, determining the stiffness ratio.

58 Finite Element Method (Overview)

Weak Formulation. For $-u_{xx} = f(x)$ on $[0, 1]$, $u(0) = u(1) = 0$: multiply by a test function $v \in H_0^1$ and integrate by parts:

$$\int_0^1 u'v' \, dx = \int_0^1 fv \, dx \quad \forall\, v \in H_0^1[0, 1].$$

Galerkin Method. Approximate $u \approx u_h = \sum_{j=1}^N c_j \phi_j(x)$, where ϕ_j are basis functions. Substituting and choosing $v = \phi_i$ gives a linear system:

$$K\mathbf{c} = \mathbf{f}, \quad K_{ij} = \int_0^1 \phi_i' \phi_j' \, dx, \quad f_i = \int_0^1 f\phi_i \, dx$$

K is the **stiffness matrix**, $\mathbf{f}$ is the **load vector**.

Piecewise Linear (Hat) Functions. On a uniform mesh $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$ with $h = 1/(N+1)$:

$$\phi_j(x) = \begin{cases} (x - x_{j-1})/h, & x_{j-1} \le x \le x_j \\ (x_{j+1} - x)/h, & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

The stiffness matrix is $K = \frac{1}{h}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{pmatrix}$, which is exactly the finite difference matrix for $-u_{xx}$!

Error Estimates. For piecewise polynomial basis of degree p:

$$\|u - u_h\|_{L^2} \le Ch^{p+1}\|u^{(p+1)}\|_{L^2}, \quad \|u' - u_h'\|_{L^2} \le Ch^p\|u^{(p+1)}\|_{L^2}$$

Example. Solve $-u'' = 1$ on $[0, 1]$, $u(0) = u(1) = 0$, by FEM with $N = 2$ $(h = 1/3)$.
Solution. One interior node at $x_1 = 1/3$, $x_2 = 2/3$. Stiffness matrix: $K = 3\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Load: $f_i = \int_0^1 1 \cdot \phi_i \, dx = 1/3$. System: $3\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$. By symmetry $c_1 = c_2 = c$: $3c = 1/3 \Rightarrow c = 1/9 \approx 0.111$. Exact: $u(1/3) = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{3} = 1/9$. The FEM solution is **exact** at the nodes (a known property for piecewise linear elements on $-u'' = f$).

59 Spectral Methods (Overview)

Expand the solution in global basis functions (Fourier modes, Chebyshev or Legendre polynomials) rather than local piecewise polynomials.

Fourier Spectral Method. For periodic problems on $[0, 2\pi]$:

$$u_N(x, t) = \sum_{|n| \le N/2} \hat{u}_n(t) e^{inx}$$

Apply the PDE in Fourier space. For $u_t = ku_{xx}$: $\hat{u}_n' = -kn^2\hat{u}_n$, giving $\hat{u}_n(t) = \hat{u}_n(0)e^{-kn^2 t}$ (exact in time for each mode).

Chebyshev Spectral Method. For non-periodic problems on $[-1, 1]$: use Chebyshev polynomials $T_n(x)$ with collocation at the Chebyshev-Gauss-Lobatto points $x_j = \cos(j\pi/N)$.

Convergence. For smooth solutions, spectral methods converge **exponentially** (faster than any polynomial rate), making them vastly more accurate per degree of freedom than finite difference or finite element methods.

Comparison.

Method	Convergence	Best For
Finite Difference	$O(h^p)$, $p = 2$ or 4	Simple domains
Finite Element	$O(h^{p+1})$ in L^2	Complex geometry
Spectral	$O(e^{-cN})$	Smooth, simple domains

60 Numerical Stability Summary

Method	PDE	Condition	Order
FTCS	$u_t = ku_{xx}$	$r \leq 1/2$	$O(k + h^2)$
BTCS	$u_t = ku_{xx}$	Unconditional	$O(k + h^2)$
Crank-Nicolson	$u_t = ku_{xx}$	Unconditional	$O(k^2 + h^2)$
Explicit	$u_{tt} = c^2 u_{xx}$	$\nu \leq 1$	$O(k^2 + h^2)$
Upwind	$u_t + au_x = 0$	$\nu \leq 1$	$O(k + h)$
Lax-Wendroff	$u_t + au_x = 0$	$\nu \leq 1$	$O(k^2 + h^2)$
FTCS	$u_t + au_x = 0$	**Unstable**	—